Someone suggests that the record of successive choices (a sequence of As and Bs) might arise from a two-state Markov chain with constant transition probabilities. Discuss, with reference to the value of $P_{n+1}(A)$ that you have found, whether this is possible.

1.1.7 Let $(X_n)_{n\geq 0}$ be a Markov chain on $\{1,2,3\}$ with transition matrix

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 2/3 & 1/3 \\ p & 1-p & 0 \end{pmatrix}.$$

Calculate $\mathbb{P}(X_n = 1 | X_0 = 1)$ in each of the following cases: (a) $p = 1/16$, (b) $p = 1/6$, (c) $p = 1/12$.

1.2 Class structure

It is sometimes possible to break a Markov chain into smaller pieces, each of which is relatively easy to understand, and which together give an understanding of the whole. This is done by identifying the communicating classes of the chain.

We say that i *leads to* j and write $i \to j$ if

$$\mathbb{P}_i(X_n = j \text{ for some } n \geq 0) > 0.$$

We say i *communicates with* j and write $i \leftrightarrow j$ if both $i \to j$ and $j \to i$.

Theorem 1.2.1. *For distinct states i and j the following are equivalent:*

(i) $i \to j$;

(ii) $p_{i_0 i_1} p_{i_1 i_2} \dots p_{i_{n-1} i_n} > 0$ *for some states* $i_0, i_1, \dots, i_n$ *with* $i_0 = i$ *and* $i_n = j$;

(iii) $p_{ij}^{(n)} > 0$ *for some* $n \geq 0$.

Proof. Observe that

$$p_{ij}^{(n)} \leq \mathbb{P}_i(X_n = j \text{ for some } n \geq 0) \leq \sum_{n=0}^{\infty} p_{ij}^{(n)}$$

which proves the equivalence of (i) and (iii). Also

$$p_{ij}^{(n)} = \sum_{i_1,\dots,i_{n-1}} p_{ii_1} p_{i_1 i_2} \dots p_{i_{n-1} j}$$

so that (ii) and (iii) are equivalent. ☐

It is clear from (ii) that $i \to j$ and $j \to k$ i
any state i. So $\leftrightarrow$ satisfies the conditions for a
and thus partitions I into *communicating class*
closed if

$$i \in C, i \to j \quad \text{imply } j \in$$

Thus a closed class is one from which there
absorbing if $\{i\}$ is a closed class. The smaller
these communicating classes. A chain or transi
single class is called *irreducible*.

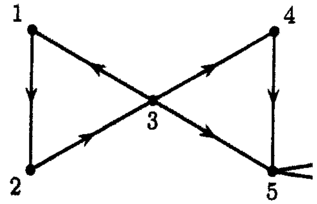
As the following example makes clear, when
the class structure of a chain is very easy to fin

Example 1.2.2

Find the communicating classes associated to t

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The solution is obvious from the diagram



the classes being $\{1,2,3\}$, $\{4\}$ and $\{5,6\}$, with

Exercises

1.2.1 Identify the communicating classes of the f

$$P = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

Which classes are closed?

It is clear from (ii) that $i \to j$ and $j \to k$ imply $i \to k$. Also $i \to i$ for any state i. So $\leftrightarrow$ satisfies the conditions for an equivalence relation on I, and thus partitions I into *communicating classes*. We say that a class C is *closed* if

$$i \in C, i \to j \quad \text{imply } j \in C.$$

Thus a closed class is one from which there is no escape. A state i is *absorbing* if $\{i\}$ is a closed class. The smaller pieces referred to above are these communicating classes. A chain or transition matrix P where I is a single class is called *irreducible*.

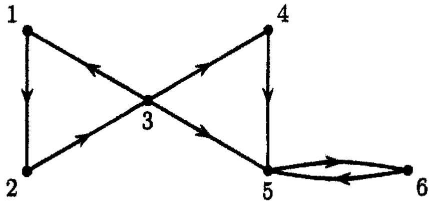
As the following example makes clear, when one can draw the diagram, the class structure of a chain is very easy to find.

Example 1.2.2

Find the communicating classes associated to the stochastic matrix

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

The solution is obvious from the diagram



the classes being $\{1,2,3\}$, $\{4\}$ and $\{5,6\}$, with only $\{5,6\}$ being closed.

Exercises

1.2.1 Identify the communicating classes of the following transition matrix:

$$P = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}.$$

Which classes are closed?

1.2.2 Show that every transition matrix on a finite state-space has at least one closed communicating class. Find an example of a transition matrix with no closed communicating class.

1.3 Hitting times and absorption probabilities

Let $(X_n)_{n\geq 0}$ be a Markov chain with transition matrix P. The *hitting time* of a subset A of I is the random variable $H^A : \Omega \to \{0, 1, 2, \dots\} \cup \{\infty\}$ given by

$$H^A(\omega) = \inf\{n \geq 0 : X_n(\omega) \in A\}$$

where we agree that the infimum of the empty set $\emptyset$ is ∞. The probability starting from i that $(X_n)_{n\geq 0}$ ever hits A is then

$$h_i^A = \mathbb{P}_i(H^A < \infty).$$

When A is a closed class, h_i^A is called the *absorption probability*. The mean time taken for $(X_n)_{n\geq 0}$ to reach A is given by

$$k_i^A = \mathbb{E}_i(H^A) = \sum_{n<\infty} n\mathbb{P}(H^A = n) + \infty\mathbb{P}(H^A = \infty).$$

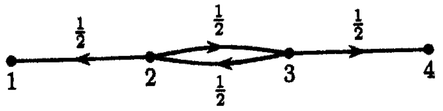
We shall often write less formally

$$h_i^A = \mathbb{P}_i(\text{hit } A), \quad k_i^A = \mathbb{E}_i(\text{time to hit } A).$$

Remarkably, these quantities can be calculated explicitly by means of certain linear equations associated with the transition matrix P. Before we give the general theory, here is a simple example.

Example 1.3.1

Consider the chain with the following diagram:



Starting from 2, what is the probability of absorption in 4? How long does it take until the chain is absorbed in 1 or 4?

Introduce

$$h_i = \mathbb{P}_i(\text{hit } 4), \quad k_i = \mathbb{E}_i(\text{time to hit } \{1, 4\}).$$

Clearly, $h_1 = 0$, $h_4 = 1$ and $k_1 = k_4 = 0$. Sup
and consider the situation after making one st
jump to 1 and with probability 1/2 we jump t

$$h_2 = \tfrac{1}{2}h_1 + \tfrac{1}{2}h_3, \quad k_2 = 1 +$$

The 1 appears in the second formula because w
step. Similarly,

$$h_3 = \tfrac{1}{2}h_2 + \tfrac{1}{2}h_4, \quad k_3 = 1 +$$

Hence

$$h_2 = \tfrac{1}{2}h_3 = \tfrac{1}{2}(\tfrac{1}{2}h_2 + \tfrac{1}{2}),$$
$$k_2 = 1 + \tfrac{1}{2}k_3 = 1 + \tfrac{1}{2}(1$$

So, starting from 2, the probability of hitting 4
absorption is 2. Note that in writing down the
we made implicit use of the Markov property,
begins afresh from its new position after the f
result for hitting probabilities.

Theorem 1.3.2. *The vector of hitting proba*
the minimal non-negative solution to the syst

$$\begin{cases} h_i^A = 1 & \textit{for} \\ h_i^A = \sum_{j\in I} p_{ij}h_j^A & \textit{for} \end{cases}$$

(Minimality means that if $x = (x_i : i \in I)$ is
for all i, then $x_i \geq h_i$ for all i.)

Proof. First we show that h^A satisfies (1.3). I
so $h_i^A = 1$. If $X_0 = i \notin A$, then $H^A \geq 1$, so by

$$\mathbb{P}_i(H^A < \infty \mid X_1 = j) = \mathbb{P}_j(H$$

and

$$h_i^A = \mathbb{P}_i(H^A < \infty) = \sum_{j\in I} \mathbb{P}_i(H^A <$$
$$= \sum_{j\in I} \mathbb{P}_i(H^A < \infty \mid X_1 = j)\mathbb{P}_i(X$$

Clearly, $h_1 = 0$, $h_4 = 1$ and $k_1 = k_4 = 0$. Suppose now that we start at 2, and consider the situation after making one step. With probability 1/2 we jump to 1 and with probability 1/2 we jump to 3. So

$$h_2 = \tfrac{1}{2}h_1 + \tfrac{1}{2}h_3, \quad k_2 = 1 + \tfrac{1}{2}k_1 + \tfrac{1}{2}k_3.$$

The 1 appears in the second formula because we count the time for the first step. Similarly,

$$h_3 = \tfrac{1}{2}h_2 + \tfrac{1}{2}h_4, \quad k_3 = 1 + \tfrac{1}{2}k_2 + \tfrac{1}{2}k_4.$$

Hence

$$\begin{aligned} h_2 &= \tfrac{1}{2}h_3 = \tfrac{1}{2}(\tfrac{1}{2}h_2 + \tfrac{1}{2}), \\ k_2 &= 1 + \tfrac{1}{2}k_3 = 1 + \tfrac{1}{2}(1 + \tfrac{1}{2}k_2). \end{aligned}$$

So, starting from 2, the probability of hitting 4 is 1/3 and the mean time to absorption is 2. Note that in writing down the first equations for h_2 and k_2 we made implicit use of the Markov property, in assuming that the chain begins afresh from its new position after the first jump. Here is a general result for hitting probabilities.

Theorem 1.3.2. *The vector of hitting probabilities $h^A = (h_i^A : i \in I)$ is the minimal non-negative solution to the system of linear equations*

$$\begin{cases} h_i^A = 1 & \text{for } i \in A \\ h_i^A = \sum_{j \in I} p_{ij} h_j^A & \text{for } i \notin A. \end{cases} \tag{1.3}$$

(*Minimality means that if $x = (x_i : i \in I)$ is another solution with $x_i \geq 0$ for all i, then $x_i \geq h_i$ for all i.*)

Proof. First we show that h^A satisfies (1.3). If $X_0 = i \in A$, then $H^A = 0$, so $h_i^A = 1$. If $X_0 = i \notin A$, then $H^A \geq 1$, so by the Markov property

$$\mathbb{P}_i(H^A < \infty \mid X_1 = j) = \mathbb{P}_j(H^A < \infty) = h_j^A$$

and

$$\begin{aligned} h_i^A = \mathbb{P}_i(H^A < \infty) &= \sum_{j \in I} \mathbb{P}_i(H^A < \infty, X_1 = j) \\ &= \sum_{j \in I} \mathbb{P}_i(H^A < \infty \mid X_1 = j)\mathbb{P}_i(X_1 = j) = \sum_{j \in I} p_{ij} h_j^A. \end{aligned}$$

Suppose now that $x = (x_i : i \in I)$ is any solution to (1.3). Then $h_i^A = x_i = 1$ for $i \in A$. Suppose $i \notin A$, then

$$x_i = \sum_{j \in I} p_{ij} x_j = \sum_{j \in A} p_{ij} + \sum_{j \notin A} p_{ij} x_j .$$

Substitute for x_j to obtain

$$\begin{aligned} x_i &= \sum_{j \in A} p_{ij} + \sum_{j \notin A} p_{ij} \Big(\sum_{k \in A} p_{jk} + \sum_{k \notin A} p_{jk} x_k \Big) \\ &= \mathbb{P}_i(X_1 \in A) + \mathbb{P}_i(X_1 \notin A, X_2 \in A) + \sum_{j \notin A} \sum_{k \notin A} p_{ij} p_{jk} x_k . \end{aligned}$$

By repeated substitution for x in the final term we obtain after n steps

$$\begin{aligned} x_i = \mathbb{P}_i(X_1 \in A) &+ \ldots + \mathbb{P}_i(X_1 \notin A, \ldots, X_{n-1} \notin A, X_n \in A) \\ &+ \sum_{j_1 \notin A} \cdots \sum_{j_n \notin A} p_{ij_1} p_{j_1 j_2} \cdots p_{j_{n-1} j_n} x_{j_n} . \end{aligned}$$

Now if x is non-negative, so is the last term on the right, and the remaining terms sum to $\mathbb{P}_i(H^A \le n)$. So $x_i \ge \mathbb{P}_i(H^A \le n)$ for all n and then

$$x_i \ge \lim_{n \to \infty} \mathbb{P}_i(H^A \le n) = \mathbb{P}_i(H^A < \infty) = h_i . \qquad \square$$

Example 1.3.1 (continued)

The system of linear equations (1.3) for $h = h^{\{4\}}$ are given here by

$$\begin{aligned} &h_4 = 1, \\ &h_2 = \tfrac{1}{2} h_1 + \tfrac{1}{2} h_3, \ h_3 = \tfrac{1}{2} h_2 + \tfrac{1}{2} h_4 \end{aligned}$$

so that

$$h_2 = \tfrac{1}{2} h_1 + \tfrac{1}{2}(\tfrac{1}{2} h_2 + \tfrac{1}{2})$$

and

$$h_2 = \tfrac{1}{3} + \tfrac{2}{3} h_1, \ h_3 = \tfrac{2}{3} + \tfrac{1}{3} h_1 .$$

The value of h_1 is not determined by the system (1.3), but the minimality condition now makes us take $h_1 = 0$, so we recover $h_2 = 1/3$ as before. Of course, the extra boundary condition $h_1 = 0$ was obvious from the beginning so we built it into our system of equations and did no minimal non-negative solutions.

In cases where the state-space is infinite it may nc down a corresponding extra boundary condition. Tł the next examples, the minimality condition is essent

Example 1.3.3 (Gamblers' ruin)

Consider the Markov chain with diagram

where $0 < p = 1 - q < 1$. The transition probabilitie

$$\begin{aligned} &p_{00} = 1, \\ &p_{i,i-1} = q, \ p_{i,i+1} = p \quad \text{for } i = 1, 2, \end{aligned}$$

Imagine that you enter a casino with a fortune of £i time, with probability p of doubling your stake and ı it. The resources of the casino are regarded as infinite limit to your fortune. But what is the probability th

Set $h_i = \mathbb{P}_i(\text{hit } 0)$, then h is the minimal non-nega

$$\begin{aligned} &h_0 = 1, \\ &h_i = p h_{i+1} + q h_{i-1}, \quad \text{for } i = 1, 2, . \end{aligned}$$

If $p \neq q$ this recurrence relation has a general solutio

$$h_i = A + B \left(\frac{q}{p} \right)^i .$$

(See Section 1.11.) If $p < q$, which is the case in mc then the restriction $0 \le h_i \le 1$ forces $B = 0$, so $h_i =$ then since $h_0 = 1$ we get a family of solutions

$$h_i = \left(\frac{q}{p} \right)^i + A \left(1 - \left(\frac{q}{p} \right)^i \right) ;$$

for a non-negative solution we must have $A \ge 0$, negative solution is $h_i = (q/p)^i$. Finally, if $p = q$ tł has a general solution

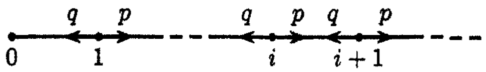
$$h_i = A + Bi$$

so we built it into our system of equations and did not have to worry about minimal non-negative solutions.

In cases where the state-space is infinite it may not be possible to write down a corresponding extra boundary condition. Then, as we shall see in the next examples, the minimality condition is essential.

Example 1.3.3 (Gamblers' ruin)

Consider the Markov chain with diagram



where $0 < p = 1 - q < 1$. The transition probabilities are

$$p_{00} = 1,$$
$$p_{i,i-1} = q, \; p_{i,i+1} = p \quad \text{for } i = 1, 2, \ldots.$$

Imagine that you enter a casino with a fortune of £i and gamble, £1 at a time, with probability p of doubling your stake and probability q of losing it. The resources of the casino are regarded as infinite, so there is no upper limit to your fortune. But what is the probability that you leave broke?

Set $h_i = \mathbb{P}_i(\text{hit } 0)$, then h is the minimal non-negative solution to

$$h_0 = 1,$$
$$h_i = ph_{i+1} + qh_{i-1}, \quad \text{for } i = 1, 2, \ldots.$$

If $p \neq q$ this recurrence relation has a general solution

$$h_i = A + B\left(\frac{q}{p}\right)^i.$$

(See Section 1.11.) If $p < q$, which is the case in most successful casinos, then the restriction $0 \leq h_i \leq 1$ forces $B = 0$, so $h_i = 1$ for all i. If $p > q$, then since $h_0 = 1$ we get a family of solutions

$$h_i = \left(\frac{q}{p}\right)^i + A\left(1 - \left(\frac{q}{p}\right)^i\right);$$

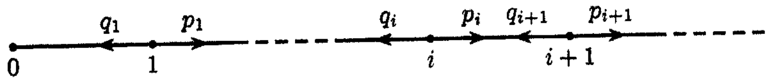
for a non-negative solution we must have $A \geq 0$, so the minimal non-negative solution is $h_i = (q/p)^i$. Finally, if $p = q$ the recurrence relation has a general solution

$$h_i = A + Bi$$

and again the restriction $0 \le h_i \le 1$ forces $B = 0$, so $h_i = 1$ for all i. Thus, even if you find a fair casino, you are certain to end up broke. This apparent paradox is called gamblers' ruin.

Example 1.3.4 (Birth-and-death chain)

Consider the Markov chain with diagram



where, for $i = 1, 2, \ldots$, we have $0 < p_i = 1 - q_i < 1$. As in the preceding example, 0 is an absorbing state and we wish to calculate the absorption probability starting from i. But here we allow p_i and q_i to depend on i.

Such a chain may serve as a model for the size of a population, recorded each time it changes, p_i being the probability that we get a birth before a death in a population of size i. Then $h_i = \mathbb{P}_i(\text{hit } 0)$ is the extinction probability starting from i.

We write down the usual system of equations

$$\begin{aligned} h_0 &= 1, \\ h_i &= p_i h_{i+1} + q_i h_{i-1}, \quad \text{for } i = 1, 2, \ldots. \end{aligned}$$

This recurrence relation has variable coefficients so the usual technique fails. But consider $u_i = h_{i-1} - h_i$, then $p_i u_{i+1} = q_i u_i$, so

$$u_{i+1} = \left(\frac{q_i}{p_i}\right) u_i = \left(\frac{q_i q_{i-1} \cdots q_1}{p_i p_{i-1} \cdots p_1}\right) u_1 = \gamma_i u_1$$

where the final equality defines γ_i. Then

$$u_1 + \ldots + u_i = h_0 - h_i$$

so

$$h_i = 1 - A(\gamma_0 + \ldots + \gamma_{i-1})$$

where $A = u_1$ and $\gamma_0 = 1$. At this point A remains to be determined. In the case $\sum_{i=0}^{\infty} \gamma_i = \infty$, the restriction $0 \le h_i \le 1$ forces $A = 0$ and $h_i = 1$ for all i. But if $\sum_{i=0}^{\infty} \gamma_i < \infty$ then we can take $A > 0$ so long as

$$1 - A(\gamma_0 + \ldots + \gamma_{i-1}) \ge 0 \quad \text{for all } i.$$

Thus the minimal non-negative solution occurs when $A = (\sum_{i=0}^{\infty}$
then

$$h_i = \sum_{j=i}^{\infty} \gamma_j \bigg/ \sum_{j=0}^{\infty} \gamma_j.$$

In this case, for $i = 1, 2, \ldots$, we have $h_i < 1$, so the population
with positive probability.

Here is the general result on mean hitting times. Recall th
$\mathbb{E}_i(H^A)$, where H^A is the first time $(X_n)_{n\ge0}$ hits A. We use the
1_B for the indicator function of B, so, for example, $1_{X_1=j}$ is th
variable equal to 1 if $X_1 = j$ and equal to 0 otherwise.

Theorem 1.3.5. *The vector of mean hitting times $k^A = (k^A :$
the minimal non-negative solution to the system of linear equatio

$$\begin{cases} k_i^A = 0 & \text{for } i \in A \\ k_i^A = 1 + \sum_{j \notin A} p_{ij} k_j^A & \text{for } i \notin A. \end{cases}$$

Proof. First we show that k^A satisfies (1.4). If $X_0 = i \in A$, then
so $k_i^A = 0$. If $X_0 = i \notin A$, then $H^A \ge 1$, so, by the Markov prope

$$\mathbb{E}_i(H^A \mid X_1 = j) = 1 + \mathbb{E}_j(H^A)$$

and

$$\begin{aligned} k_i^A = \mathbb{E}_i(H^A) &= \sum_{j \in I} \mathbb{E}_i(H^A 1_{X_1=j}) \\ &= \sum_{j \in I} \mathbb{E}_i(H^A \mid X_1 = j)\mathbb{P}_i(X_1 = j) = 1 + \sum_{j \notin A} p_{ij} k_j^A. \end{aligned}$$

Suppose now that $y = (y_i : i \in I)$ is any solution to (1.4). Then k_i^A
for $i \in A$. If $i \notin A$, then

$$\begin{aligned} y_i &= 1 + \sum_{j \notin A} p_{ij} y_j \\ &= 1 + \sum_{j \notin A} p_{ij} \left(1 + \sum_{k \notin A} p_{jk} y_k\right) \\ &= \mathbb{P}_i(H^A \ge 1) + \mathbb{P}_i(H^A \ge 2) + \sum_{j \notin A} \sum_{k \notin A} p_{ij} p_{jk} y_k. \end{aligned}$$

By repeated substitution for y in the final term we obtain after n

$$y_i = \mathbb{P}_i(H^A \ge 1) + \ldots + \mathbb{P}_i(H^A \ge n) + \sum_{j_1 \notin A} \cdots \sum_{j_n \notin A} p_{ij_1} p_{j_1 j_2} \cdots p_j$$

Thus the minimal non-negative solution occurs when $A = \left(\sum_{i=0}^{\infty}\gamma_i\right)^{-1}$ and then

$$h_i = \sum_{j=i}^{\infty}\gamma_j \Big/ \sum_{j=0}^{\infty}\gamma_j.$$

In this case, for $i = 1, 2, \dots$, we have $h_i < 1$, so the population survives with positive probability.

Here is the general result on mean hitting times. Recall that $k_i^A = \mathbb{E}_i(H^A)$, where H^A is the first time $(X_n)_{n\geq 0}$ hits A. We use the notation 1_B for the indicator function of B, so, for example, $1_{X_1=j}$ is the random variable equal to 1 if $X_1 = j$ and equal to 0 otherwise.

Theorem 1.3.5. *The vector of mean hitting times $k^A = (k^A : i \in I)$ is the minimal non-negative solution to the system of linear equations*

$$\begin{cases} k_i^A = 0 & \text{for } i \in A \\ k_i^A = 1 + \sum_{j\notin A} p_{ij}k_j^A & \text{for } i \notin A. \end{cases} \tag{1.4}$$

Proof. First we show that k^A satisfies (1.4). If $X_0 = i \in A$, then $H^A = 0$, so $k_i^A = 0$. If $X_0 = i \notin A$, then $H^A \geq 1$, so, by the Markov property,

$$\mathbb{E}_i(H^A \mid X_1 = j) = 1 + \mathbb{E}_j(H^A)$$

and

$$\begin{aligned} k_i^A = \mathbb{E}_i(H^A) &= \sum_{j\in I}\mathbb{E}_i(H^A 1_{X_1=j}) \\ &= \sum_{j\in I}\mathbb{E}_i(H^A \mid X_1 = j)\mathbb{P}_i(X_1 = j) = 1 + \sum_{j\notin A} p_{ij}k_j^A. \end{aligned}$$

Suppose now that $y = (y_i : i \in I)$ is any solution to (1.4). Then $k_i^A = y_i = 0$ for $i \in A$. If $i \notin A$, then

$$\begin{aligned} y_i &= 1 + \sum_{j\notin A} p_{ij}y_j \\ &= 1 + \sum_{j\notin A} p_{ij}\Big(1 + \sum_{k\notin A} p_{jk}y_k\Big) \\ &= \mathbb{P}_i(H^A \geq 1) + \mathbb{P}_i(H^A \geq 2) + \sum_{j\notin A}\sum_{k\notin A} p_{ij}p_{jk}y_k. \end{aligned}$$

By repeated substitution for y in the final term we obtain after n steps

$$y_i = \mathbb{P}_i(H^A \geq 1) + \ldots + \mathbb{P}_i(H^A \geq n) + \sum_{j_1\notin A} \cdots \sum_{j_n\notin A} p_{ij_1}p_{j_1j_2}\cdots p_{j_{n-1}j_n}y_{j_n}.$$

So, if y is non-negative,

$$y_i \geq \mathbb{P}_i(H^A \geq 1) + \dots + \mathbb{P}_i(H^A \geq n)$$

and, letting $n \to \infty$,

$$y_i \geq \sum_{n=1}^{\infty} \mathbb{P}_i(H^A \geq n) = \mathbb{E}_i(H^A) = x_i. \qquad \square$$

Exercises

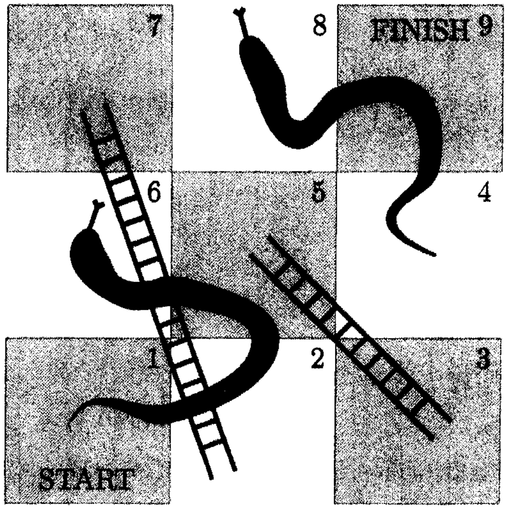
1.3.1 Prove the claims (a), (b) and (c) made in example (v) of the Introduction.

1.3.2 A gambler has £2 and needs to increase it to £10 in a hurry. He can play a game with the following rules: a fair coin is tossed; if a player bets on the right side, he wins a sum equal to his stake, and his stake is returned; otherwise he loses his stake. The gambler decides to use a bold strategy in which he stakes all his money if he has £5 or less, and otherwise stakes just enough to increase his capital, if he wins, to £10.

Let $X_0 = 2$ and let X_n be his capital after n throws. Prove that the gambler will achieve his aim with probability 1/5.

What is the expected number of tosses until the gambler either achieves his aim or loses his capital?

1.3.3 A simple game of 'snakes and ladders' is played on a board of nine squares.



At each turn a player tosses a fair coin a: according to whether the coin lands heads of a ladder you climb to the top, but if you slide down to the tail. How many turns on ε the game?

What is the probability that a player whc will complete the game without slipping ba

1.3.4 Let $(X_n)_{n\geq 0}$ be a Markov chain on {(bilities given by

$$p_{01} = 1, \quad p_{i,i+1} + p_{i,i-1} = 1, \quad p_{i,i+1} =$$

Show that if $X_0 = 0$ then the probability th

1.4 Strong Markov

In Section 1.1 we proved the Markov proper m, conditional on $X_m = i$, the process af i. Suppose, instead of conditioning on X_m process to hit state i, at some random time process after time H? What if we replacec time, for example $H - 1$? In this section we times at which a version of the Markov proj include H but not $H - 1$; after all, the pı straight to i, so it does not simply begin afı

A random variable $T : \Omega \to \{0, 1, 2, \dots\}$ (if the event $\{T = n\}$ depends only on $X_0,$ Intuitively, by watching the process, you kn If asked to stop at T, you know when to stc

Examples 1.4.1

(a) The *first passage time*

$$T_j = \inf\{n \geq 1 : X,$$

is a stopping time because

$$\{T_j = n\} = \{X_1 \neq j, \dots, X_n$$

(b) The first hitting time H^A of Section 1.3

At each turn a player tosses a fair coin and advances one or two places according to whether the coin lands heads or tails. If you land at the foot of a ladder you climb to the top, but if you land at the head of a snake you slide down to the tail. How many turns on average does it take to complete the game?

What is the probability that a player who has reached the middle square will complete the game without slipping back to square 1?

1.3.4 Let $(X_n)_{n\geq 0}$ be a Markov chain on $\{0,1,\dots\}$ with transition probabilities given by

$$p_{01}=1, \quad p_{i,i+1}+p_{i,i-1}=1, \quad p_{i,i+1}=\left(\frac{i+1}{i}\right)^2 p_{i,i-1}, \quad i\geq 1\,.$$

Show that if $X_0=0$ then the probability that $X_n\geq 1$ for all $n\geq 1$ is $6/\pi^2$.

1.4 Strong Markov property

In Section 1.1 we proved the Markov property. This says that for each time m, conditional on $X_m=i$, the process after time m begins afresh from i. Suppose, instead of conditioning on $X_m=i$, we simply waited for the process to hit state i, at some random time H. What can one say about the process after time H? What if we replaced H by a more general random time, for example $H-1$? In this section we shall identify a class of random times at which a version of the Markov property does hold. This class will include H but not $H-1$; after all, the process after time $H-1$ jumps straight to i, so it does not simply begin afresh.

A random variable $T:\Omega\to\{0,1,2,\dots\}\cup\{\infty\}$ is called a *stopping time* if the event $\{T=n\}$ depends only on $X_0,X_1,\dots,X_n$ for $n=0,1,2,\dots$. Intuitively, by watching the process, you know at the time when T occurs. If asked to stop at T, you know when to stop.

Examples 1.4.1

(a) The *first passage time*

$$T_j=\inf\{n\geq 1: X_n=j\}$$

is a stopping time because

$$\{T_j=n\}=\{X_1\neq j,\dots,X_{n-1}\neq j, X_n=j\}.$$

(b) The first hitting time H^A of Section 1.3 is a stopping time because

$$\{H^A=n\}=\{X_0\notin A,\dots,X_{n-1}\notin A, X_n\in A\}.$$