MATH 285, HW#7, SOLUTIONS, SPRING 07

1. ξ_i are iid $P(\xi_i = 1) = p$, $P(\xi_i = -1) = q$, $p < 1/2$

$$S_n = x + \sum_{i=1}^{n} \xi_i$$

a) $M_n = S_n + (q-p)n$ is a martingale wrt. $\sigma(S_0, \ldots, S_n) = \mathcal{F}_n$

i) $M_n \in \mathcal{F}_n$. Let $\mathcal{G}_n = \sigma(M_0, \ldots, M_n)$.
Then $M_n \in \mathcal{G}_n$ by definition of $\mathcal{G}_n$. $\mathcal{G}_n = \mathcal{F}_n$ since at any time n S_n and M_n differ by a deterministic constant. So $M_n \in \mathcal{F}_n$.

ii) $E[|M_n|] < \infty$ for all n.

$$|M_n| \le |x| + \left|\sum_{i=1}^{n} \xi_i\right| + |q-p|n \le |x| + n + 2n$$

so $E[|M_n|] < |x| + 3n < \infty$ for any n.
M_n is integrable.

iii) $E[M_{n+1} | \mathcal{F}_n] = E[S_n + \xi_{n+1} + (q-p)(n+1) | \mathcal{F}_n] = S_n + E[\xi_{n+1}] + (q-p)(n+1) = S_n + (p-q) + (q-p)(n+1) = S_n + (q-p)n = M_n$.

The second equality is true because $S_n \in \mathcal{F}_n$, $\xi_{n+1} \perp\!\!\!\perp \mathcal{F}_n$, $(q-p)(n+1)$ is a constant and conditional expectation is a linear operator.

b) $M_n = \left(\frac{q}{p}\right)^{S_n}$ is a martingale wrt

$\mathcal{F}_n = \sigma(M_0, \ldots, M_n)$.

i) $M_n \in \mathcal{F}_n$ by definition of $\mathcal{F}_n$

ii) $E|M_n| < \infty$ for all n. Since $q > p$ $\frac{q}{p} > 1$.

$|S_n| \leq |x| + \left|\sum_{i=1}^{n} \xi_i\right| \leq x + \sum_{i=1}^{n} |\xi_i| \leq x + n$

(Note $x > 0$ so $|x| = x$).

Since $\frac{q}{p} > 1$ $\left|\left(\frac{q}{p}\right)^{S_n}\right| \leq \left(\frac{q}{p}\right)^{|S_n|} \leq \left(\frac{q}{p}\right)^{x+n}$

so $E|M_n| < \left(\frac{q}{p}\right)^{x+n} < \infty$ for all n.

iii) $E[M_{n+1} | \mathcal{F}_n] = E\left[\left(\frac{q}{p}\right)^{S_n}\left(\frac{q}{p}\right)^{\xi_{n+1}} \Big| \mathcal{F}_n\right]$

$= \left(\frac{q}{p}\right)^{S_n} E\left[\left(\frac{q}{p}\right)^{\xi_{n+1}}\right]$ since $S_n \in \mathcal{F}_n$ and

$\xi_{n+1} \perp\!\!\!\perp \mathcal{F}_n$ $= \left(\frac{q}{p}\right)^{S_n}\left(\frac{q}{\cancel{p}}\cancel{p} + \frac{p}{\cancel{q}}\cancel{q}\right) =$

$\left(\frac{q}{p}\right)^{S_n} = M_n$

c) $T = \inf\{n \geq 0 : S_n = 0 \text{ or } b\}$ is a

stopping time for $\mathcal{F}_n = \sigma(S_0, \ldots, S_n)$

$\{T = k\} = \{S_0 \notin \{0, b\}, \ldots, S_{k-1} \notin \{0, b\},$

$S_k \in \{0, b\}\} \in \mathcal{F}_k$ since k is arbitrary the result follows.

d) Let M_n be as in b). We are assuming that $T < \infty$ a.s. Also whenever $n \le T$ $|M_n| \le \left(\frac{q}{p}\right)^{b+x}$ so by Thm 4.1.1 part ii) $EM_T = EM_0$.

$$EM_0 = \left(\frac{q}{p}\right)^x = EM_T$$

$$EM_T = \left(\frac{q}{p}\right)^b P(S_T = b) + P(S_T = 0)$$

We have that $P(S_T = b) + P(S_T = 0) = 1$.

We get that

$$P(S_T = 0) = \frac{\left(\frac{q}{p}\right)^x - \left(\frac{q}{p}\right)^b}{1 - \left(\frac{q}{p}\right)^b}$$

$\{S_T = 0\} = \{S_n$ reaches 0 before $b\}$.

2. $X = \{ X_t , t \geq 0 \}$ is a standard 1 dim'l BM. Want to show that

$Y = \{ Y_t , t \geq 0 \} = \{ \frac{X_{at}}{\sqrt{a}} , t \geq 0 \}$ is also a standard 1 dim'l BM.

i) $Y_0 = X_0 = 0$ as

ii) for any $0 = t_0 < t_1 < \cdots < t_n$

$\{ X_{at_1} - X_{at_0}, X_{at_2} - X_{at_1}, \ldots, X_{at_n} - X_{at_{n-1}} \}$

are independent. Scaling by $\frac{1}{\sqrt{a}}$ does not change independence so

$$\left\{ \frac{1}{\sqrt{a}} (X_{at_1} - X_{at_0}), \ldots, \frac{1}{\sqrt{a}} (X_{at_n} - X_{at_{n-1}}) \right\}$$

$= \{ Y_{t_1} - Y_{t_0}, \ldots, Y_{t_n} - Y_{t_{n-1}} \}$ are independent.

iii) $E(Y_t - Y_s) = \frac{1}{\sqrt{a}} E(X_{at} - X_{as}) = 0$

$$Var (Y_t - Y_s) = Var \left(\frac{1}{\sqrt{a}} (X_{at} - X_{as}) \right)$$

$$= \frac{1}{a} Var (X_{at} - X_{as}) = \frac{1}{a} a(t-s)$$

So $Y_t - Y_s$ is normal rv with mean 0 and variance $(t-s)$.

iv) $t \longrightarrow \frac{X_{at}}{\sqrt{a}}$ is continuous :

$$t \longrightarrow at \longrightarrow X_{at} \longrightarrow \frac{X_{at}}{\sqrt{a}}$$

since it is a composition of continuous mappings.

Note : $Y_t - Y_s$ is normal because $X_{at} - X_{as}$ is normal because X is a standard 1 dim'l BM.

$\{ Y_t, \; t \geq 0 \} = \{ \frac{X_{at}}{\sqrt{a}}, \; t \geq 0 \}$ is a standard 1 dim'l BM.

3. $X = \{X_t, t \geq 0\}$ is a standard 1 dim'l BM. Compute $P(X_2 > 0 \mid X_1 > 0)$

$$P(X_2 > 0 \mid X_1 > 0) = \frac{P(X_2 > 0, X_1 > 0)}{P(X_1 > 0)}$$

$= 2\,P(X_2 > 0, X_1 > 0)$ Since $P(X_1 > 0) = 1/2$ by symmetry of BM

$P(X_2 > 0, X_1 > 0) = P((X_2 - X_1) + X_1 > 0, X_1 > 0)$ Let $Y = (X_2 - X_1)$

Then $Y \sim N(0,1)$ and $Y \perp\!\!\!\perp X_1$.

$P(X_2 > 0, X_1 > 0) = P(Y > -X_1, X_1 > 0)$

The joint density of (X_1, Y) is

$$f(x,y) = \frac{1}{2\pi} \exp\left(-\frac{(x^2+y^2)}{2}\right)$$

So

$$P(Y > -X_1, X_1 > 0) = \int_0^\infty \int_{-x}^\infty f(x,y)\,dy\,dx$$

$$= \int_0^{\infty}\int_{-x}^{\infty} \frac{1}{2\pi} \exp\left(-\frac{(x^2+y^2)}{2}\right) dy\, dx$$

switching to polar coordinates

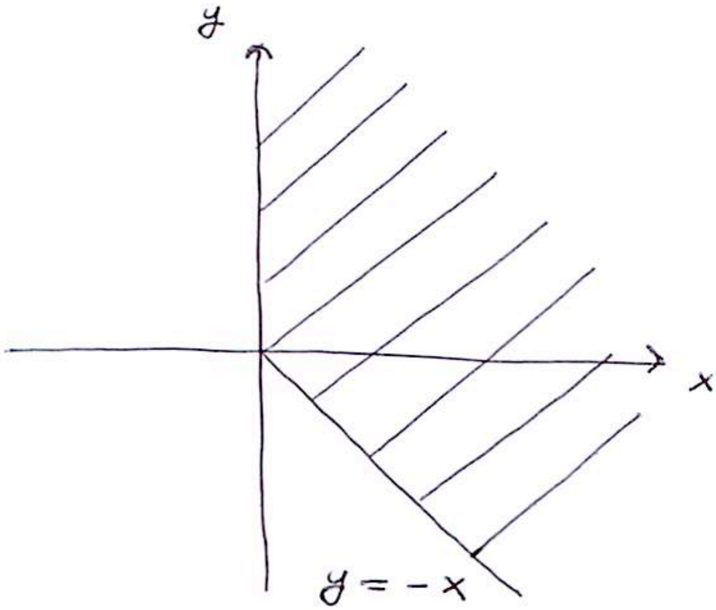
$$= \frac{1}{2\pi}\int_{-\pi/4}^{\pi/2}\int_0^{\infty} \exp(-r^2/2)\, r\, dr\, d\theta$$

$u = r^2 \qquad du = 2r\,dr \qquad r\,dr = \frac{du}{2}$

$$= \frac{1}{4\pi}\int_{-\pi/4}^{\pi/2}\int_0^{\infty} \exp(-u/2)\, du\, d\theta$$

$$= \frac{1}{4\pi}\int_{-\pi/4}^{\pi/2} -2\exp(-u/2)\Big|_0^{\infty} d\theta$$

$$= \frac{1}{2\pi}\int_{-\pi/4}^{\pi/2} 1\, d\theta = \frac{1}{2\pi}\left(\pi/2 + \pi/4\right) = \frac{3}{8}$$



So

$P(X_2 > 0 \mid X_1 > 0)$

$= 3/4$