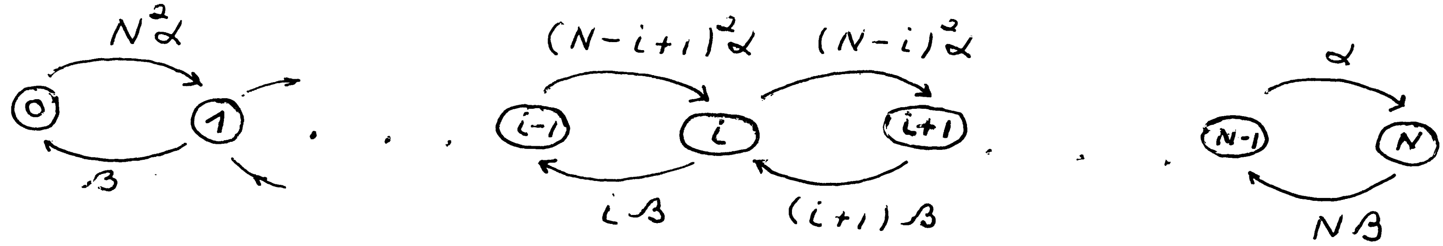
MATH 285, HW # 6, SOLUTIONS, SPRING 07

1. Let X_t denote the number of molecules of type AB at time t.

$$S = \{0, 1, 2, \ldots, N\}$$

We have the following diagram:



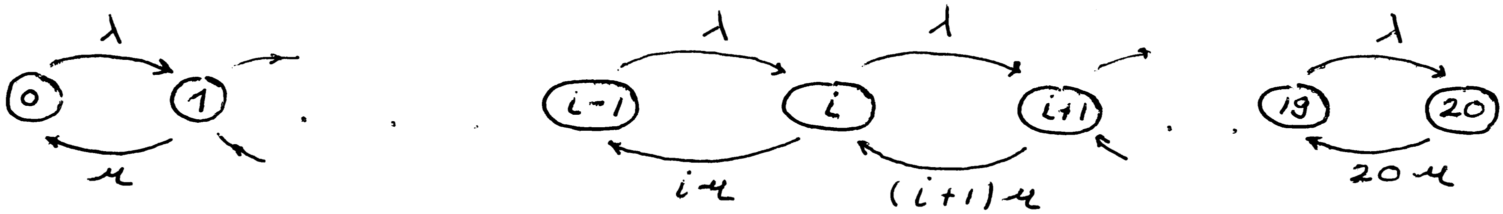
Since if at time t we have i molecules of type AB, we have $N-i$ A molecules and $N-i$ B molecules. A and B molecules can combine in $(N-i)^2$ ways to produce an AB molecule, each reaction occurs at rate α, so the total rate of going from i to $i+1$ is $(N-i)^2\alpha$. On the other hand each one of the i AB molecules decays independently at rate β, so the total rate of going from i to $i-1$ is $i\beta$. Q matrix is given ~~by~~

$$Q_{ij} = \begin{cases} (N-i)^2\alpha & \text{if } j = i+1,\ i \geq 0,\ i < N \\ i\beta & \text{if } j = i-1,\ i > 0,\ i \leq N \\ -(i\beta + (N-i)^2\alpha) & \text{if } j = i \quad 0 \leq i \leq N \\ 0 & \text{otherwise} \end{cases}$$

2. Let X_t = # of cars in the lot at time t

$S = \{0, 1, 2, \dots, 20\}$

We have the following diagram:



Since the total arrival rate is λ and each car leaves independently at rate μ.

a)

$$Q_{ij} = \begin{cases} \lambda & \text{if } j = i+1 \\ i\mu & \text{if } j = i-1 \\ -(\lambda + i\mu) & \text{if } j = i, \quad i < 20 \\ -20\mu & \text{if } j = i = 20 \\ 0 & \text{otherwise} \end{cases}$$

b) π is a stationary distribution if π is a distribution and $\pi' Q = 0$. So equations are:

i) $-\pi_0 \lambda + \mu \pi_1 = 0$

ii) $\pi_{i-1} \lambda - (i\mu + \lambda)\pi_i + (i+1)\mu \pi_{i+1} = 0$

for $i = 1, \dots, 19$

iii) $\lambda \pi_{19} - 20\mu \pi_{20} = 0 \qquad i = 20$

iv) $\pi_0 + \pi_1 + \dots + \pi_{20} = 1$

c) $$\pi_i = \left(\frac{\lambda}{\mu}\right)^i \frac{1}{i!} \pi_0$$

where
$$\pi_0 = \frac{1}{\sum_{i=0}^{20} \left(\frac{\lambda}{\mu}\right)^i \frac{1}{i!}}$$

We show this ~~by~~ induction.

From part b) $\pi_1 = \frac{\lambda}{\mu} \pi_0$.

Assume that the result holds for $1, \dots, i$ and we want to show that it holds for $i+1$.

From b)
$$\pi_{i+1} = \frac{(i\mu + \lambda)\pi_i - \pi_{i-1}\lambda}{(i+1)\mu}$$

$$= \frac{(i\mu + \lambda)\left(\frac{\lambda}{\mu}\right)^i \frac{1}{i!} \pi_0 - \left(\frac{\lambda}{\mu}\right)^{i-1} \frac{1}{(i-1)!} \pi_0 \lambda}{(i+1)\mu}$$

$$= \frac{\frac{1}{(i-1)!}\left(\frac{\lambda}{\mu}\right)^{i-1} \pi_0 \left((i\mu + \lambda)\frac{1}{i}\left(\frac{\lambda}{\mu}\right) - \lambda\right)}{(i+1)\mu}$$

$$= \frac{\left(\frac{\lambda}{\mu}\right)^{i-1} \pi_0 \frac{\lambda^2}{i\mu} \frac{1}{(i-1)!}}{(i+1)\mu} = \left(\frac{\lambda}{\mu}\right)^{i+1} \frac{1}{(i+1)!} \pi_0$$

We need to check that this is consistent with equation iii). From iii)

$$\pi_{20} = \left(\frac{\lambda}{\mu}\right) \frac{1}{20} \pi_{19} = \left(\frac{\lambda}{\mu}\right) \frac{1}{20} \left(\frac{\lambda}{\mu}\right)^{19} \frac{1}{19!} \pi_0$$

$$= \left(\frac{\lambda}{\mu}\right)^{20} \frac{1}{20!} \pi_0 .$$

$$\pi_0 = \frac{1}{\sum_{i=0}^{20} \left(\frac{\lambda}{\mu}\right) \frac{1}{i!}}$$ is choosen to normalize the sum $\sum_{i=0}^{20} \pi_i$ to 1.

$$\sum_{i=0}^{20} \left(\frac{\lambda}{\mu}\right)^i \frac{1}{i!} \pi_0 = \pi_0 \sum_{i=0}^{20} \left(\frac{\lambda}{\mu}\right)^i \frac{1}{i!} = 1 .$$

d) This continuous time MC is irreducible since its discrete time skeleton is irreducible. Since S is finite it is non-explosive. It is also positive recurrent. By Thm 3.6.2 $\lim_{t \to \infty} p_{ij}(t) = \pi_j$ for any j. Hence the long run probability that the chain is in state 20 (parking lot is full) is

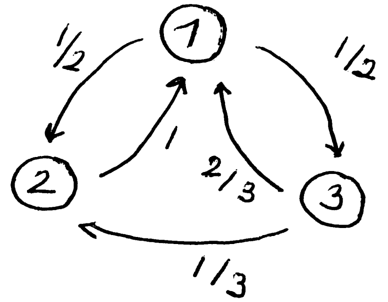
$$\pi_{20} = \left(\frac{\lambda}{\mu}\right)^{20} \frac{\pi_0}{20!}$$

$\boxed{\Sigma}$

3. $Q = \begin{pmatrix} -2 & 1 & 1 \\ 4 & -4 & 0 \\ 2 & 1 & -3 \end{pmatrix}$

Discrete time skeleton:

$$P = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1 & 0 & 0 \\ 2/3 & 1/3 & 0 \end{pmatrix}$$



Discrete time skeleton is irreducible, therefore continuous time MC is irreducible.

To find a stationary distribution we need to find π' so that $\pi' Q = 0$.

This has a unique solution

$$\pi = \begin{pmatrix} 3/5 \\ 1/5 \\ 1/5 \end{pmatrix}$$

This continuous time Markov chain is non explosive and positive recurrent since the state space is finite. Again we can apply Thm 3.6.2 to conclude

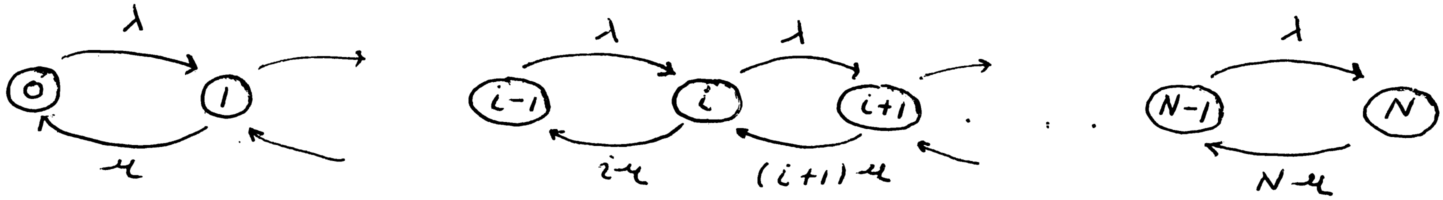
that $\lim_{t \to \infty} P_{12}(t) = 1/5$

$\boxtimes$

Problem 4.

Let X_t = # of buses in operation at time t.

$S = \{0, \ldots, N\}$



We seek a stationary distribution satisfying detailed balance equations:

$$\pi_i Q_{i,i+1} = \pi_{i+1} Q_{i+1,i}$$

$$\Longleftrightarrow \quad \pi_i \lambda = \pi_{i+1} (i+1)\mu$$

$$\Longleftrightarrow \quad \pi_{i+1} = \frac{\lambda}{\mu} \frac{1}{i+1} \pi_i$$

$$\Longleftrightarrow \quad \pi_i = \left(\frac{\lambda}{\mu}\right)^i \frac{1}{i!} \pi_0$$

$$\sum_{i=0}^{N} \left(\frac{\lambda}{\mu}\right)^i \frac{1}{i!} < \infty$$

Set $$\pi_0 = \frac{1}{\sum_{i=0}^{N} \left(\frac{\lambda}{\mu}\right)^i \frac{1}{i!}}$$

This makes $(\pi_0, \ldots, \pi_N)$ a probability distribution.

A distribution which satisfies detailed balance is necessarily invariant.

$$\pi' Q = 0$$

Again this chain is irreducible, possitive recurrent and non-explosive and by Thm 3.6.2 $\lim_{t \to \infty} p_{ij}(t) = \pi_j$ is an equilibrium distribution.

Note: This method could have been used to solve problem 2 which is essentially the same $(N = 20)$.