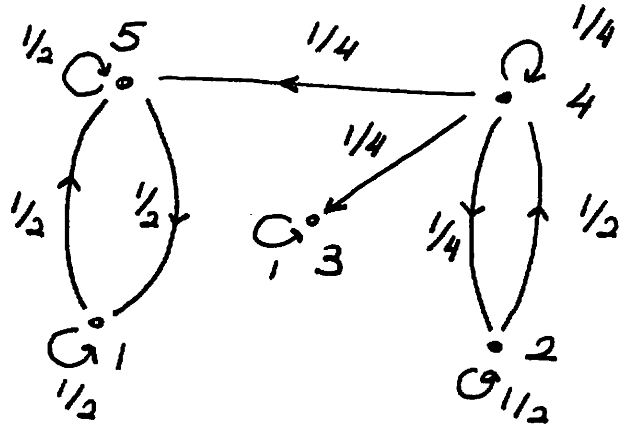
MATH 285, HW 4, SOLUTIONS, SPRING 2007

Problem 1.2.1



By HW 3 this Markov chain has 3 communicating classes $\{1,5\}$, $\{2,4\}$, $\{3\}$. $\{1,5\}$, $\{3\}$ are closed and therefore recurrent by Thm 1.5.6. (Thm 1.5.6: Every finite closed class is recurrent).

By Thm 1.5.4 and 1.5.3 recurrence and transience are class properties and in particular every class/state is either recurrent or transient. So states 1, 5, 3 are recurrent.

$\{1,4\}$ is not a closed class. Therefore by contrapositive of Thm 1.5.5 $\{1,4\}$ is not a recurrent class. So $\{1,4\}$ is a transient class and 1, 4 are transient states. ∎

7 Problem 1.8.2 (a)

To find an invariant distribution we need to find a solution to the following vector-matrix equation: $\pi P = \pi$

$$(\pi_1, \pi_2, \pi_3) \begin{pmatrix} 0 & 1 & 0 \\ 0 & 2/3 & 1/3 \\ 1/16 & 15/16 & 0 \end{pmatrix} = (\pi_1, \pi_2, \pi_3)$$

subject to $\pi_1 + \pi_2 + \pi_3 = 1$, $\pi_i \geq 0 \ \forall i$.

This vector-matrix equation has a unique solution, subject to the constraint, $(\pi_1, \pi_2, \pi_3) = (1/65, 48/65, 16/65)$. This solution is the unique invariant measure.

This Markov chain has a single communicating class (it is possible to get from any state to any other state in finitely many steps), i.e. it is irreducible.

It is also recurrent by Thm. 1.5.6.

and therefore since the State Space is finite it is positive recurrent.

Using MATLAB we see that p_{22} and p_{22}^2 is are both greater than zero.

Hence $\{n \geq 1 : p_{22}^n > 0\}$ has no common divisor other than 1. We conclude that state 2 and therefore all other states are aperiodic. So the Markov chain is aperiodic.

Now we can apply Theorem 1.8.3 to conclude that $p_{ij}^{(n)} \longrightarrow \pi_j$ as $n \longrightarrow \infty$ for all i, j.

Therefore $\pi = (\pi_1, \pi_2, \pi_3)$ is a steady state distribution. π_j is the long run probability that MC is in state j.

So long run probabilities that MC is in state 1, 2, 3 respectively are $1/65$, $48/65$, $16/65$

See next page.

4 Theorem 1.8.3 ensures that the results we got will agree with results we would obtain in Excercise 1.1.7 a).

We could have done this explicitly.

From section 1 we know that

$p^n_{11} = a\lambda_1^n + b\lambda_2^n + c\lambda_3^n$ where $\lambda_1, \lambda_2, \lambda_3$ are eigenvalues of P. Using MATLAB we see that $\lambda_1 = -1/4$, $\lambda_2 = -1/12$, $\lambda_3 = 1$.

So $p^n_{11} = a(-1/4)^n + b(-1/12)^n + c$.

$p^0_{11} = 1$ (since MC starts at 1), $p_{11} = p^2_{11} = 0$.

We obtain a, b, c by solving the system:

$$a + b + c = 1$$

$$-1/4\, a - 1/12\, b + c = 0$$

$$1/16\, a + 1/144\, b + c = 0.$$

We find that $a = -2/5$, $b = 18/13$ and $c = 1/65$.

So $p^{n}_{11} = -2/5 \, (-1/4)^{n} + 18/13 \, (-1/12)^{n} + 1/65$

and

$$\lim_{n \to \infty} p^{n}_{11} = 1/65 = \pi_1$$

Similarly we would find that

for any i, j $\quad p^{n}_{ij} \longrightarrow \pi_j$ as $n \longrightarrow \infty$.

```
  To get started, select MATLAB Help or Demos from the Help menu.

>> P=[0,1,0;0,2/3,1/3;1/16,15/16,0]
```

MATLAB for second part of 1.8.2

```
P =

       0              1              0
       0              2/3            1/3
       1/16          15/16           0

>> eig(P)

ans =

      -1/4
      -1/12
       1

>> P^2

ans =

       0              2/3            1/3
       1/48         109/144          2/9
       0             11/16           5/16

>> S=[1,1,1;-1/4,-1/12,1;1/16,1/144,1]

S =

       1              1              1
      -1/4           -1/12           1
       1/16           1/144          1
```

S corresponds to the system we need to solve to obtain a, b, c.

```
>> inv(S)

ans =

      -2/5          -22/5           24/5
      18/13          54/13         -72/13
       1/65          16/65          48/65

>> u=[1,0,0]'
```

$u = (p^0, p^1_{11}, p^2_{11})$

```
u =

       1
       0
       0

>> inv(S)*u
```

7


```
ans =

      -2/5
      18/13
       1/65

>>
```

ADDED PROBLEM

i)

The Markov Chain in Excercise 1.3.4 has a single communicating class, the State Space $S = \{0, 1, 2, \ldots\}$. (It is possible to go from each state to any other state in finitely many steps)

We showed that $P_0(X_n \geq 1 \text{ for all } n \geq 1) = \frac{6}{\pi^2}$.

So $P_0(T_0 < \infty) = 1 - P_0(T_0 = \infty) =$

$$1 - P_0(X_n \geq 1 \text{ for all } n \geq 1) = 1 - \frac{6}{\pi^2} < 1.$$

So $\{0\}$ is a transient state.

Recurrence and transience are class properties.

Therefore all states are transient.

ii) We want to show that

$$P_i(X_n \longrightarrow \infty \text{ as } n \longrightarrow \infty) = 1 \quad \forall i \geq 1.$$

Let $A = \{X_n \longrightarrow \infty \text{ as } n \longrightarrow \infty\}$.

Let $A_k = \{X_n \geq k \text{ for all suff. large } n\}$ $k \in \mathbb{N}$.

Then $A = \bigcap_{k=0}^{\infty} A_k$.

9/ Also $A_{k+1} \subset A_k$, so $P_i(A) = \lim_{k \to \infty} P_i(A_k)$.

It suffices to show that $P_i(A_k) = 1$ for all k. Showing that $P_i(A_k) = 1$ is equivalent to showing that $P_i(A_k^c) = 0$ for any k.

$$A_k^c = \{ X_n < k \text{ for infinitely many } n \}$$

$$= \bigcup_{j=0}^{k-1} \{ X_n = j \text{ for infinitely many } n \}$$

So

$$P_i(A_k^c) = P_i\left(\bigcup_{j=0}^{k-1} \{ X_n = j \text{ infinitely often} \} \right)$$

$$P_i(A_k^c) \leq \sum_{j=0}^{k-1} P_i(\{ X_n = j \text{ i.o.} \}).$$

To establish that $P_i(A_k^c) = 0$ it is enough to show that for any j

$$P_i(\{ X_n = j \text{ i.o.} \}) = 0.$$

To show this we will use Stong Markov Property.

Remember that this MC is transient

so $P_j(X_n = j \text{ i.o}) = 0$ for all j.

$$P_i(X_n = j \text{ i.o}) = P_j(X_n = j \text{ i.o.}) \cdot$$

$P_i(T_j < \infty)$ by the Stang Markov Property.

$= 0.$ See next page for details.

So $P_i(A_k^c) = 0$ for all k.

Therefore $P_i(A_k) = 1$ for all k, and

$$P_i(A) = \lim_{k \to \infty} P_i(A_k) = 1 \text{ for any } i$$

since i was arbitrary.

11)

Note:

$$P_i (X_n = j \quad i.o.)$$

$$= P_i (X_n = j \quad i.o, \; T_j < \infty) + P(X_n = j \; i.o., \; T_j = \infty) = P_i (X_n = j \quad i.o., \; T_j < \infty)$$

$$= P_i (X_{T_j + n} = j \quad i.o., \; T_j < \infty, \; X_{T_j} = j)$$

$$= P_i (X_{T_j + n} = j \quad i.o. \mid T_j < \infty, \; X_{T_j} = j) \cdot P_i (T_j < \infty, \; X_{T_j} = j) =$$ by Strong Markov Property

$$P_j (X_n = j \quad i.o) \, P_i (T_j < \infty)$$