MATH 285, SPRING 2007
HOMEWORK 7, due Tuesday, June 12, 2007

This assignment is *optional.* If you hand it in, the score you receive will replace your lowest score on other assignments.

1. Consider a biased simple random walk on the integers with probability $p < 1/2$ of moving to the right by one and probability $q = 1 - p$ of moving to the left by one, at each step. Let S_n denote the position of the random walk at time n, for $n = 0, 1, 2, \ldots$. Then we may represent S_n by

$$S_n = x + \sum_{i=1}^{n} \xi_i, \quad n = 0, 1, \ldots,$$

where x is the initial state, $\{\xi_i\}_{i=1}^{\infty}$ are i.i.d. random variables with $P(\xi_i = +1) = p$ and $P(\xi_i = -1) = q$ for each i. Suppose that $0 < x < b$ where x and b are positive integers.

(a) Show that $M_n = S_n + (q-p)n, n = 0, 1, 2, \ldots$ is a martingale with respect to the filtration generated by $S_n, n = 0, 1, 2, \ldots$.

(b) Show that $M_n = (q/p)^{S_n}, n = 0, 1, 2, \ldots,$ is a martingale with respect to its own filtration.

(c) Let $T = \inf\{n \geq 0 : S_n = 0 \text{ or } b\}$. Show that T is a stopping time relative to the filtration generated by $\{S_n, n = 0, 1, 2, \ldots\}$.

(d) Assume that T is finite with probability one. Use the optional stopping theorem and your answer to part (b) to find the probability that the random walk reaches 0 before b.

2. Suppose that $X = \{X_t, t \geq 0\}$ is standard one-dimensional Brownian motion. Fix $a > 0$. Prove (using the definition) that $\{X(at)/\sqrt{a}, t \geq 0\}$ is also a standard one-dimensional Brownian motion.

3. Let $X = \{X_t, t \geq 0\}$ be a standard one-dimensional Brownian motion. Compute the conditional probability,

$$P(X_2 > 0 | X_1 > 0).$$