MATH 285 HW 6 SPRING 2007

May 16, 2007

Homework is due Monday, June 4.

1. A chemical solution initially contains N molecules of type A and an equal number of molecules of type B. A reversible reaction occurs between type A and B molecules in which a molecule of type A bonds with a molecule of type B to form a new compound AB. Suppose that in any small time interval of length h, any particular unbonded A molecule will react with any particular unbonded B molecule with probability $\alpha h + o(h)$, where α is a positive reaction rate. Suppose also that in any small time interval of length h, any particular AB molecule dissociates into its A and B constituents with probability $\beta h + o(h)$, where β is a positive reaction rate. Let $X(t)$ denote the number of AB molecules at time t. Model X as a birth-death process and specify its infinitesimal matrix Q.

2. A shopping center parking lot has twenty spaces. Cars arrive at the lot according to a Poisson process with rate $\lambda > 0$. If a car arrives at the lot and there is an empty space, it takes the space. If a car arrives and there are no empty spaces, the car moves on to another lot and never returns. Any parked car remains in its space for an exponentially distributed amount of time with mean $1/\mu$ where μ is a positive finite number. Assume that the times that different cars remain parked are independent and that these times are independent of the Poisson arrival process. Take as given that the number of cars in the lot at time t is a continuous time Markov chain.

(a) Determine the infinitesimal transition probabilities (that is the Q matrix) for this Markov chain. (It may be helpful to draw a diagram of the states with the infinitesimal rates marked on it).

(b) Write down the equations for a stationary distribution $\pi = (\pi_0, \ldots, \pi_{20})$ of this Markov chain.

(c) Solve the equations in (b) for π_i, $i = 1, \ldots, 20$. (You may put your answer in terms of π_0 and give a series expression for π_0.)

(d) Find an expression for the long run probability that the lot is full (in terms of π_0).

3. Consider a continuous time Markov chain with Q matrix

$$Q = \begin{pmatrix} -2 & 1 & 1 \\ 4 & -4 & 0 \\ 2 & 1 & -3 \end{pmatrix} \tag{1}$$

Is this chain irreducible? Find a stationary distribution for this chain. What is $\lim_{t\to\infty} P_{12}(t)$?

4. Problem 3.7.1 from the Text by Norris.