MATH 285, HW 5, SOLUTIONS, SPRING 07

Problem 1.9.1 P is assumed to be irreducible throughout the problem.

b)

$$P = \begin{pmatrix} 0 & p & 1-p \\ 1-p & 0 & p \\ p & 1-p & 0 \end{pmatrix}$$

P has a unique stationary distribution $\pi = (1/3, 1/3, 1/3)$. According to the Thm. 1.9.3 the MC corresponding to P will be reversible iff π, P are in detailed balance. ie $\pi_i p_{ij} = \pi_j p_{ji}$.

This is the case if $p_{ij} = p_{ji}$ for all i, j ie $P = P^T$. $P = P^T$ iff $p = 1/2$. So when $p = 1/2$ P is reversible, otherwise it is irreversible.

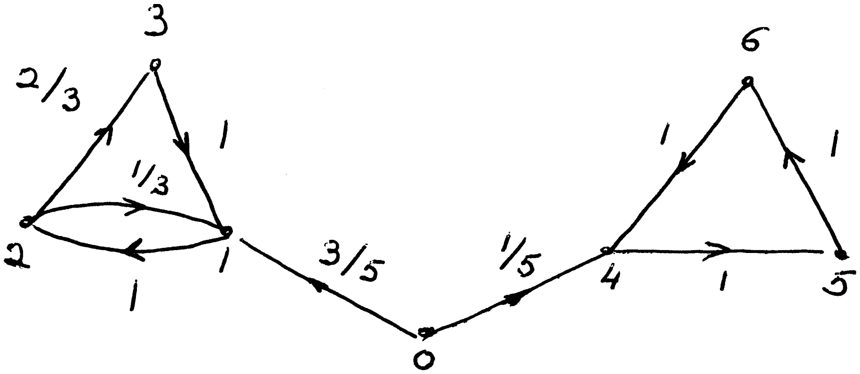
e) $p_{ij} = p_{ji}$ and S is finite.

Let $\lambda_i = \frac{1}{|S|}$ where $|S|$ = cardinality of S.

Then $\lambda = (\lambda_i)_{i \in S}$ is an invariant distribution for P and P, λ are in detailed balance.

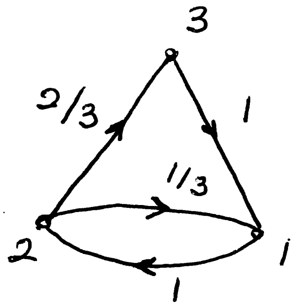
According to Thm 1.9.3 P is reversible.

Problem 1.10.1



We want to show that starting at 1 the long run proportion of time spent in 2 is 3/8.

$\{1,2,3\}$ is a closed communicating class so starting at 1 we only need to consider the MC $(X_n)_{n\geq 0}$ on these 3 states, starting at 1.



$$P = \begin{pmatrix} 0 & 1 & 0 \\ 1/3 & 0 & 2/3 \\ 1 & 0 & 0 \end{pmatrix}$$

P is irreducible.

Let $V_2(n) = \sum_{k=0}^{n-1} 1_{\{X_k = 2\}}$.

According to Thm 1.10.2

$$\mathbb{P}\left(\frac{V_2(n)}{n} \longrightarrow \frac{1}{m_2} \text{ as } n \longrightarrow \infty \right) = 1$$

where $m_2 = E_2(T_2)$.

So the long run proportion of time spent in 2 is $\frac{1}{m_2}$.

Next Page $\Rightarrow$

We want to compute $m_2 = E_2(T_2)$.

Starting at 2 we will return to this state in either 2 or 3 steps.

$$m_2 = E_2(T_2) = 2\,P_2(T_2 = 2) + 3\,P_2(T_2 = 3)$$

$$= 2\,P_2(X_1 = 1, X_2 = 2) + 3\,P_2(X_1 = 3, X_2 = 1, X_3 = 2)$$

$$= 2\,p_{21}p_{12} + 3\,p_{23}p_{31}p_{12} = 2 \cdot 1/3 + 3 \cdot 2/3$$

$$= 8/3.$$

So the long run proportion of time spent in 2 is $3/8$.

Problem 3

Prove that $T = \inf\{n > 10 : X_n = 6\}$ is a stopping time for the MC in example 1.2.2 starting at 1.

All we need to show is that the event $\{T = m\}$ depends only on $X_0, \ldots, X_m$ for any $m \in \mathbb{N}$.

$$\{T = m\} = \begin{cases} \emptyset & \text{if } m \leq 10 \\ \{X_0 = 1, X_2 \in S, \ldots, X_{10} \in S, X_{11} \neq 6, \ldots \\ \ldots, X_{m-1} \neq 6, X_m = 6\} & \text{if } m \geq 11 \end{cases}$$

In both cases $\{T = m\}$ is determined by $X_0, \ldots, X_m$.

T is a stopping time.

Problem 4 State $S = \{ \text{Coin } 1, \text{Coin } 2 \}$

HMM $(\pi_1, \pi_2) = (1/2, 1/2)$

$$a = \begin{pmatrix} .7 & .3 \\ .4 & .6 \end{pmatrix}$$

$b_1(H) = P(H \mid \text{Coin } 1) = 1/2$

$b_1(T) = P(H \mid \text{Coin } 2) = 1/2$

$b_2(H) = P(H \mid \text{Coin } 2) = 1/4$

$b_2(T) = P(T \mid \text{Coin } 2) = 3/4$

a) The observed sequence $O^* = (O_0^*, O_1^*, O_2^*, O_3^*, O_4^*, O_5^*) = (H, H, T, T, T, H)$. We want to find out the most likely state sequence to generate this output.

Use Viterbi algorithm

i) $\delta_0(i) = \pi_i \, b_i(O_0^*)$

ii) Recursion $\delta_{t+1}(j) = \max\limits_{i=1}^{N} [a_{ij} \, b_j(O_{t+1}^*) \, \delta_t(i)]$

$$\psi_{t+1}(j) = \operatorname*{argmax}_{i=1}^{N} \delta_{t+1}(j)$$

i) Initialization $(t = 0, O_0^* = H)$

$\delta_0(1) = \pi_1 b_1(H) = 1/2 \cdot 1/2 = 1/4$

$\delta_0(2) = \pi_2 b_2(H) = 1/2 \cdot 1/4 = 1/8$

ii) Recursion $t=1$ $o_1 = H$

$$\delta_1(1) = \max_{i=1,2} [a_{i1}\, b_1(H)\, \delta_0(i)]$$

$$\delta_0(1) a_{11} = (.25)(.7) = .175$$

$$\delta_0(2) a_{21} = (.125)(.4) = .05$$

so $\delta_1(1) = \delta_0(1)\, a_{11}\, b_1(H) = .0875$

$$\psi_1(1) = 1$$

$$\delta_1(2) = \max_{i=1,2} [a_{i2}\, b_2(H)\, \delta_0(i)]$$

$$a_{12}\, b_2(H)\, \delta_0(1) = .3 \cdot .25 \cdot .25 = .0187$$

$$a_{22}\, b_2(H)\, \delta_0(2) = .6 \cdot .25 \cdot .125 = .0187$$

so $\delta_1(2) = .0187$ and $\psi_1(2) = 1$ or 2
(since we have a tie)

$\delta_1(1) = .0875$

$\psi_1(1) = 1$

$\delta_1(2) = .0187$

$\psi_1(2) = 1$ or 2

$t=2$ $o_2^* = T$

$$\delta_2(1) = \max_{i=1,2} [a_{i1}\, b_1(T)\, \delta_1(i)]$$

proceeding the same way as when $t = 1$ we find that

$\delta_2(1) = .03 \qquad \psi_2(1) = 1$

$\delta_2(2) = .019 \qquad \psi_2(2) = 2$

$t = 3 \qquad O_3^* = T$

$\delta_3(1) = .011 \qquad \psi_3(1) = 1$

$\delta_3(2) = .009 \qquad \psi_3(2) = 2$

$t = 4 \qquad O_4^* = T$

$\delta_4(1) = .0037 \qquad \psi_4(1) = 1$

$\delta_4(2) = .0039 \qquad \psi_4(2) = 2$

$t = 5 \qquad O_5^* = H$

$\delta_5(1) = .0013 \qquad \psi_5(1) = 1$

$\delta_5(2) = .0006 \qquad \psi_5(2) = 2$

Work backwards to find the optimal q

$$q_5^* = \arg\max_{i=1,2} [\delta_5(i)] = 1$$

$$q_4^* = \psi_5(q_5^*) = \psi_5(1) = 1$$

$$q_3^* = \psi_4(q_4^*) = \psi_4(1) = 1$$

$$q_2^* = \psi_3(q_3^*) = \psi_3(1) = 1$$

$$q_1^* = \psi_2(q_2^*) = \psi_2(1) = 1$$

$$q_0^* = \psi_1(q_1^*) = \psi_1(1) = 1$$

Therefore the most likely state sequence to generate the output HHTTTH

is $(q_5^*, q_4^*, q_3^*, q_2^*, q_1^*, q_0^*) = (1,1,1,1,1,1)$

ie HMM is in Coin 1.

b) We want to compute

$$P(\text{Coin } 2 \mid O^*) = \frac{P(\text{Coin } 2, O^*)}{P(O^*)}$$

$$= \frac{\pi_2\, b_2(H)\, a_{22}\, b_2(H)\, a_{22}\, b_2(T)\, a_{22}\, b_2(T)\, a_{22}\, b_2(T)\, a_{22}\, b_2(H)}{P(O^*)}$$

$$= \frac{.5\,(.6)^5\,(.25)^3\,(.75)^3}{P(O^*)}$$

All we need to do is compute $P(O^*)$.

For this we use the Forward Algorithm.

i) $\alpha_0(1) = \pi_1 b_1(H) = 1/4$

$\alpha_0(2) = \pi_2 b_2(H) = .125$

ii) Recursion

$t = 1 \quad O_1^* = H$

$\alpha_1(1) = (\alpha_0(1) a_{11} + \alpha_0(2) a_{21}) b_1(H)$

$= (.175 + .05) .5 = .1125$

$\alpha_1(2) = (\alpha_0(1) a_{12} + \alpha_0(2) a_{22}) b_1(H)$

$= .0375$

So $(\alpha_1(1), \alpha_1(2)) = (.1125, .0375)$

Proceeding the same way we find that

$(\alpha_2(1), \alpha_2(2)) = (.047, .042)$

$(\alpha_3(1), \alpha_3(2)) = (.025, .029)$

$(\alpha_4(1), \alpha_4(2)) = (.015, .019)$

$(\alpha_5(1), \alpha_5(2)) = (.0089, .0039)$

$P(O^*) = \alpha_5(1) + \alpha_5(2) = .0128$

Then $P(\text{Coin } 2 \mid O^*) = \dfrac{.5\,(.6)^5(.25)^3(.75)^3}{.0128}$

$= .02$

The probability that the observed sequence was generated entirely by Coin 2 is .02