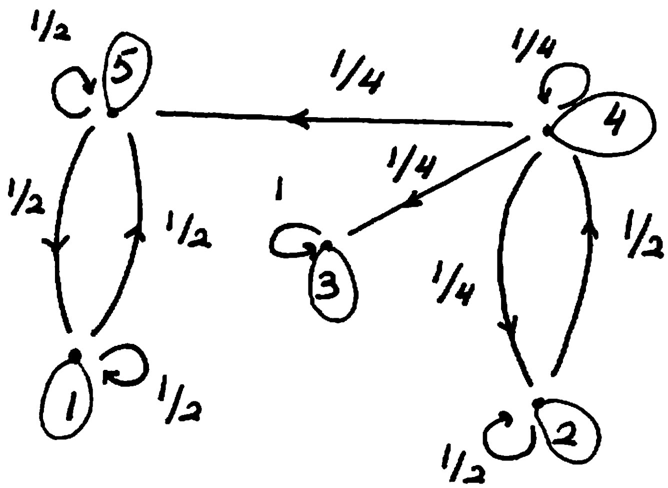
MATH 285, HW3, SOLUTIONS, SPRING 2007

Problem 1.2.1 Identify the communicating classes of the following matrix.

$$P = \begin{pmatrix} 1/2 & 0 & 0 & 0 & 1/2 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1/4 & 1/4 & 1/4 & 1/4 \\ 1/2 & 0 & 0 & 0 & 1/2 \end{pmatrix}$$

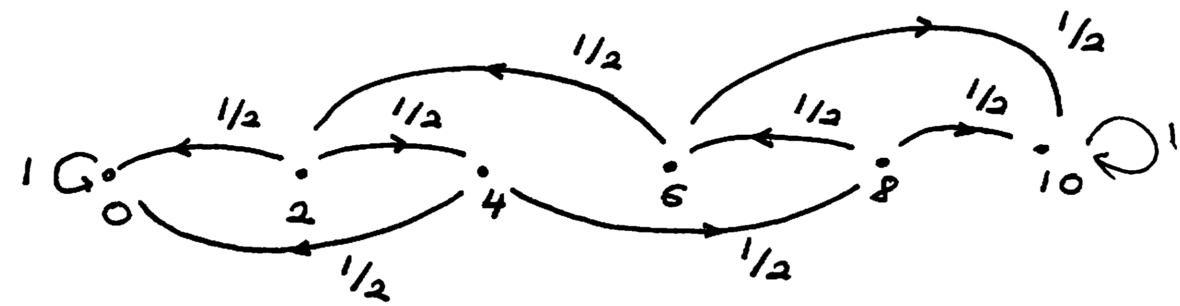


Communicating classes are

$\{1,5\}$, $\{3\}$, $\{4,2\}$.

$\{1,5\}$, $\{3\}$ are closed.

Problem 1.3.2



i) X_n is the capital after n throws. $X_0 = 2$.
We want to show that the player will increase his capital to 10 with probability $1/5$.
Let $H^{10} = \inf\{n \geq 0 : X_n = 10\}$.
Let $h_i = P_i(H^{10} < \infty)$. We want to compute h_2.
$h_2 = P_2(\text{hit } 10) = P_2(H^{10} < \infty)$ is the probability that starting at 2 player will reach 10 before reaching 0. According to Thm 1.3.2 the vector of hitting probabilities $h^{10} = \{h_i^{10} : i \in \{0,2,4,6,8,10\}\}$, is the minimal nonnegative solution to the following system: $h_0 = h_0$, $h_{10} = 1$, $h_2 = \frac{1}{2} h_4$,

$h_4 = \frac{1}{2} h_8$, $h_6 = \frac{1}{2} h_2 + \frac{1}{2}$, $h_8 = \frac{1}{2} h_6 + \frac{1}{2}$.

Since we are seeking the minimal solution set $h_0 = 0$
(also, we know that $P_0(H^{10} < \infty) = 0$ so must have $h_0 = 0$)
Then this system has a unique finite solution

By finite we mean $h_i < \infty \ \forall i$. $\Rightarrow$ Next page

$(h_0, h_2, h_4, h_6, h_8, h_{10}) = (0, 1/5, 2/5, 3/5, 4/5, 1)$.

The uniqueness ensures minimality.

Therefore $h_2 = 1/5$. This completes the first part.

ii) The expected number of tosses until the game ends is $E_2(H^A)$ where $A = \{0, 10\}$.

Let $k_i = E_i(H^A)$. Then we are looking for k_2 and according to Thm. 1.3.5 the vector of mean hitting times $k^A = (k_i^A : i \in \{0, 2, 4, 6, 8, 10\})$ is the minimal nonnegative solution to the system

$$k_0 = 0, \quad k_{10} = 0$$

$$k_2 = 1 + 1/2\, k_4, \quad k_4 = 1 + 1/2\, k_8$$

$$k_6 = 1 + 1/2\, k_2, \quad k_8 = 1 + 1/2\, k_6$$

This system has a unique finite solution

$$(k_0, k_2, k_4, k_6, k_8, k_{10}) = (0, 2, 2, 2, 2, 0).$$

Again uniqueness ensures minimality.

The expected number of tosses until the game ends is 2, i.e. $k_2 = 2$ $\blacksquare$

Problem 1.3.4 $(X_n)_{n\geq 1}$ Markov on $\mathbb{N}$

$p_{01} = 1$ $\quad p_{i,i+1} + p_{i,i-1} = 1$, $p_{i,i+1} = \left(\frac{i+1}{i}\right)^2 p_{i,i-1}$

$i \geq 1$. We want to show that if $X_0 = 0$

then $P(X_n \geq 1 \ \forall n \geq 1) = \frac{6}{\pi^2}$.

Pf: Let $h_i = P_i(\text{hit } 0)$, $i \in \mathbb{N}$.

Then $P(X_n \geq 1 \ \forall n \geq 1) = 1 - h_1$.

We want to compute h_1.

This is a Birth-and-death chain.

Let $p_i = p_{i,i+1}$ and $q_i = p_{i,i-1}$

Then $p_i + q_i = 1$ and $p_i = \left(\frac{i+1}{i}\right)^2 q_i$

Solving these 2 equations we get that

$$p_i = \frac{\left(\frac{i+1}{i}\right)^2}{1 + \left(\frac{i+1}{i}\right)^2}$$ and

$$q_i = \frac{1}{1 + \left(\frac{i+1}{i}\right)^2}$$

According to the page 15 of the textbook the minimal nonnegative solution to the system

$$\begin{cases} h_0 = 1 \\ h_i = p_i h_{i+1} + q_i h_{i-1} & i = 1, 2, \dots \end{cases}$$

is finite ~~occurs~~ when $\sum_{i=0}^{\infty} \gamma_i < \infty$

and the solution is

$$h_i = \frac{\sum_{j=i}^{\infty} \gamma_j}{\sum_{j=0}^{\infty} \gamma_j},$$

where

$$\gamma_i = \frac{\prod_{j=1}^{i} q_j}{\prod_{j=1}^{i} p_j}.$$

In our problem $\gamma_i = \dfrac{1}{(i+1)^2}$

and the condition $\sum_{i=0}^{\infty} \gamma_i < \infty$ is satisfied since $\sum_{i=0}^{\infty} \dfrac{1}{(i+1)^2} = \dfrac{\pi^2}{6}$

according to Euler's result.

Therefore $$h_1 = \frac{\sum_{i=1}^{\infty} \frac{1}{(i+1)^2}}{\sum_{i=0}^{\infty} \frac{1}{(i+1)^2}}$$

$$= \frac{\sum_{i=0}^{\infty} \frac{1}{(i+1)^2} - 1}{\sum_{i=0}^{\infty} \frac{1}{(i+1)^2}} = \frac{\frac{\pi^2}{6} - 1}{\frac{\pi^2}{6}} = 1 - \frac{6}{\pi^2}$$

So $P(X_n \geq 1 \ \forall n \geq 1) = 1 - h_1$

$$= 1 - \left(1 - \frac{6}{\pi^2}\right) = \frac{6}{\pi^2}$$ $\blacksquare$