MATH 285, HW 2 SOLUTIONS, SPRING 07

PROBLEM 1.1.2

$(X_n)_{n \geq 0}$ is Markov (λ, P). If $Y_n = X_{kn}$ show that $(Y_n)_{n \geq 0}$ is Markov (λ, P^k).

Pf:

i) Show that Y_n has initial distribution λ.

$$P(Y_0 = i) = P(X_{k \cdot 0} = i) = P(X_0 = i) = \lambda_i$$

ii) Show that Y_n satisfies Markov property

ie

$$P(Y_{n+1} = i_{n+1} \mid Y_n = i_n, \ldots, Y_0 = i_0)$$
$$\overset{?}{=} P(Y_{n+1} = i_{n+1} \mid Y_n = i_n).$$

$$P(Y_{n+1} = i_{n+1} \mid Y_n = i_n, \ldots, Y_0 = i_0) =$$
$$P(X_{k \cdot (n+1)} = i_{n+1} \mid X_{k \cdot n} = i_n, \ldots, X_0 = i_0) =$$
$$P(X_{k \cdot (n+1)} = i_{n+1} \mid X_{kn} = i_n) =$$
$$P(Y_{n+1} = i_{n+1} \mid Y_n = i_n)$$

The second equality is by Markov property for X_n. This proves that $(Y_n)_{n \geq 0}$ is Markov.

iii). Show that P^k is the transition probability matrix for Y_n ie $P(Y_{n+1} = i_{n+1} \mid Y_n = i_n) = P^k_{i_n i_{n+1}}$

$P(Y_{n+1} = i_{n+1} \mid Y_n = i_n) = P(X_{k(n+1)} = i_{n+1} \mid X_{kn} = i_n) = P(X_{kn+k} = i_{n+1} \mid X_{kn} = i_n)$

$= P^k_{i_n i_{n+1}}$ by Theorem 1.1.3

This completes the proof.

$(Y_n)_{n \geq 0}$ is Markov

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Note that for any Markov chain

$$\mathcal{P}(X_{n+m} = i_{n+m} \mid X_n = i_n, \ldots, X_0 = i_0) =$$

$$\sum_{i_{n+1},\ldots,i_{n+m-1} \in S} \mathcal{P}(X_{n+m} = i_{n+m}, X_{n+m-1} = i_{n+m-1}, \ldots$$

$$\ldots, X_{n+1} = i_{n+1} \mid X_n = i_n, \ldots, X_0 = i_0)$$

By MP for X_n

$$\overset{\downarrow}{=} \sum_{i_{n+1},\ldots,i_{n+m-1} \in S} \mathcal{P}(X_{n+m} = i_{n+m}, X_{n+m-1} = i_{n+m-1}, \ldots$$

$$\ldots, X_{n+1} = i_{n+1} \mid X_n = i_n) =$$

$$\mathcal{P}(X_{n+m} = i_{n+m} \mid X_n = i_n)$$

Second solution to problem 1.1.2

By Thm 1.1.1 in order to show that $(Y_n)_{n\geq 0}$ is Markov (λ, P^k) it is enough to show that for all $i_0, \ldots, i_n \in S$

$$P(Y_0 = i_0, \ldots, Y_n = i_n) = \lambda_{i_0} \cdot P^k_{i_0 i_1} \cdot \ldots \cdot P^k_{i_{n-1} i_n}.$$

$$P(Y_0 = i_0, \ldots, Y_n = i_n) = P(X_0 = i_0, \ldots, X_{kn} = i_n)$$

$$= P(X_{kn} = i_n \mid X_{k(n-1)} = i_{n-1}) \cdot \ldots \cdot P(X_k = i_1 \mid X_0 = i_0) \cdot P(X_0 = i_0)$$

$\uparrow$ by MP for X_n

$$= P^k_{i_{n-1}, i_n} \cdot \ldots \cdot P^k_{i_0 i_1} \cdot \lambda_{i_0}$$

$\uparrow$ by Thm 1.1.3

By Thm 1.1.1 $(Y_n)_{n\geq 0}$ is Markov.

Problem 1.1.3

$(Z_i)_{i \geq 0}$ are iid such that $P(Z_i = 1) = p$ and $P(Z_i = 0) = 1 - p$. Set $S_0 = 0$, $S_n = \sum_{i=1}^{n} Z_i$.

In each of the following cases determine if $(X_n)_{n \geq 0}$ is a Markov chain.

a) $X_n = Z_n$ $\quad$ $\{X_n\}$ is Markov

$$P(X_{n+1} = i_{n+1} \mid X_n = i_n, \ldots, X_0 = i_0) =$$

$$P(Z_{n+1} = i_{n+1} \mid Z_n = i_n, \ldots, Z_0 = i_0)$$

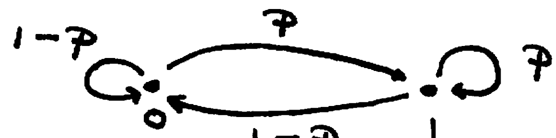
$$= P(Z_{n+1} = i_{n+1}) \text{ since } Z_i\text{'s are i.i.d.}$$

(for details see next page)

$$= P(Z_{n+1} = i_{n+1} \mid Z_n = i_n)$$

$$= P(X_{n+1} = i_{n+1} \mid X_n = i_n)$$

$S = \{0, 1\}$



$$P = \begin{pmatrix} 1-p & p \\ 1-p & p \end{pmatrix}$$

d) $X_n = S_n$ $\{S_n\}$ is Markov

Let $A = \{ S_{n-1} = i_{n-1}, \ldots, S_0 = i_0 \}$

$$P(S_{n+1} = i_{n+1} \mid S_n = i_n, A) =$$

$$P(S_n + Z_{n+1} = i_{n+1} \mid S_n = i_n, A) =$$

$$\frac{P(S_n + Z_{n+1} = i_{n+1}, S_n = i_n, A)}{P(S_n = i_n, A)} =$$

$$\frac{P(Z_{n+1} = i_{n+1} - i_n, S_n = i_n, A)}{P(S_n = i_n, A)} \underset{\substack{\uparrow \\ \text{Independence of} \\ Z_i\text{'s}}}{=}$$

$$\frac{P(Z_{n+1} = i_{n+1} - i_n)\, P(S_n = i_n, A)}{P(S_n = i_n, A)} =$$

$$P(Z_{n+1} = i_{n+1} - i_n) =$$

$$\frac{P(Z_{n+1} = i_{n+1} - i_n)\, P(S_n = i_n)}{P(S_n = i_n)} =$$

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$$= \frac{P(Z_{n+1} = i_{n+1} - i_n, \ S_n = i_n)}{P(S_n = i_n)} =$$

$$= \frac{P(S_n + Z_{n+1} = i_{n+1}, \ S_n = i_n)}{P(S_n = i_n)}$$

$$= P(S_{n+1} = i_{n+1} \mid S_n = i_n)$$

This establishes MP.

State Space $S = \mathbb{N}$

$$P_{ij} = \begin{cases} 1-p & \text{if } i = j \\ p & \text{if } j = i+1 \\ 0 & \text{otherwise} \end{cases}$$

∎

c) $X_n = S_0 + \dots + S_n)$ ($n \geq 0$) is not Markov

We need to show that for some n and $i_0, \dots, i_n$

$$P(X_{n+1} = i_{n+1} \mid X_n = i_n, \dots, X_0 = i_0) \neq P(X_{n+1} = i_{n+1} \mid X_n = i_n)$$

Note that $X_n = nZ_1 + (n-1)Z_2 + \dots + 2Z_{n-1} + Z_n$

We are going to show that for $p \neq 1/2$

$$P(X_4 = 5 \mid X_3 = 3, X_2 = 1, X_1 = 0) \neq P(X_4 = 5 \mid X_3 = 3)$$

$$X_4 = 4Z_1 + 3Z_2 + 2Z_3 + Z_4$$

$$X_3 = 3Z_1 + 2Z_2 + Z_3$$

$$X_2 = 2Z_1 + Z_2$$

$$X_1 = Z_1$$

$$\{X_4 = 5\} = \{Z_1 = Z_4 = 1, Z_2 = Z_3 = 0\} \cup \{Z_1 = Z_4 = 0, Z_2 = Z_3 = 1\}$$

$$\{X_3 = 3\} = \{Z_1 = 1, Z_2 = Z_3 = 0\} \cup \{Z_1 = 0, Z_2 = Z_3 = 1\}$$

$$\{X_2 = 1\} = \{Z_1 = 0, Z_2 = 1\}, \quad \{X_1 = 0\} = \{Z_1 = 0\}$$

$$P(X_4 = 5 \mid X_3 = 3, X_2 = 1, X_1 = 0)$$

$$= \frac{P(X_4 = 5, X_3 = 3, X_2 = 1, X_1 = 0)}{P(X_3 = 3, X_2 = 1, X_1 = 0)} = \text{Next page}$$

$$= \frac{P(Z_1 = 0,\ Z_2 = Z_3 = 1,\ Z_4 = 0)}{P(Z_1 = 0,\ Z_2 = Z_3 = 1)} = \frac{p^2(1-p)^2}{p^2(1-p)} = (1-p)$$

On the other hand

$$P(X_4 = 5 \mid X_3 = 3) = \frac{P(X_4 = 5,\ X_3 = 3)}{P(X_3 = 3)} =$$

$$\frac{P(\{Z_1 = Z_4 = 1,\ Z_2 = Z_3 = 0\} \cup \{Z_1 = Z_4 = 0,\ Z_3 = Z_2 = 1\})}{P(\{Z_1 = 1,\ Z_2 = Z_3 = 0\} \cup \{Z_1 = 0,\ Z_2 = Z_3 = 1\})}$$

$$= \frac{2p^2(1-p)^2}{p(1-p)^2 + p^2(1-p)} = \frac{2p^2(1-p)^2}{p(1-p)} = 2p(1-p)$$

if $p \neq 1/2$ $\quad 2p(1-p) \neq (1-p)$

If $p = 1/2$ by doing similar computation we see that

$$P(X_6 = 7 \mid X_5 = 5,\ X_4 = 4,\ X_3 = 3,\ X_2 = 2,\ X_1 = 1) = p = 1/2$$

while

$$P(X_6 = 7 \mid X_5 = 5) = 3p(1-p) = \frac{3}{4}$$

So $\{X_n\}$ is not Markov in that case.

d) $X_n = (S_n, S_0 + \dots + S_n)$; $\{X_n\}$ is Markov

Let $Y_n = S_0 + \quad + S_n$

$(S_0, Y_0) = (i_0, j_0)$

$$P((S_{n+1}, Y_{n+1}) = (i_{n+1}, j_{n+1}) \mid (S_n, Y_n) = (i_n, j_n), \dots)$$

$$= P((S_n + Z_{n+1}, Y_n + S_n + Z_{n+1}) = (i_{n+1}, j_{n+1}) \mid (S_n, Y_n) = (i_n, j_n), \dots) = P((i_n + Z_{n+1}, j_n + i_n + Z_{n+1}) = (i_{n+1}, j_{n+1}) \mid (S_n, Y_n) = (i_n, j_n), \dots) =$$

$$P((Z_{n+1}, Z_{n+1}) = (i_{n+1} - i_n, j_{n+1} - j_n - i_n))$$

$$= P(Z_{n+1} = i_{n+1} - i_n = j_{n+1} - j_n - i_n)$$

since Z_{n+1} is independent of $(S_i, Y_i)_{i=0}^n$.

For details see part b).

Now $P((Z_{n+1}, Z_{n+1}) = (i_{n+1} - i_n, j_{n+1} - j_n - i_n))$

$$= P((Z_{n+1}, Z_{n+1}) = (i_{n+1} - i_n, j_{n+1} - j_n - i_n) \mid (S_n, Y_n) = (i_n, j_n)) = P((S_{n+1}, Y_{n+1}) = (i_{n+1}, j_{n+1}) \mid (S_n, Y_n) = (i_n, j_n))$$

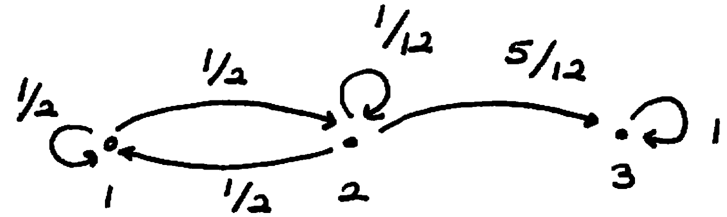
This establishes MP.

The state space for X_n is

$$S = \{ (i,j) \in \mathbb{N} \times \mathbb{N} : j \geq i \}$$

$$P_{(i,j)(k,l)} = \begin{cases} 1-p & \text{if } i = k \quad l = j+k \\ p & \text{if } k = i+1 \quad l = j+k \\ 0 & \text{otherwise} \end{cases}$$

Problem 1.1.6



$$P = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 5/12 \\ 0 & 0 & 1 \end{pmatrix}$$

i) The probability that the octopus is in state 1 just before the n+1 st trial is $P(X_n = 1) = P^n_{11}$

We want to obtain a general formula for P^n_{11}.

By discussion on page 8 of the textbook

$P^n_{11} = a(\lambda_1)^n + b(\lambda_2)^n + c(\lambda_3)^n$ where $\lambda_1, \lambda_2, \lambda_3$ are the eigenvalues of P. Performing computation manually or using MATHLAB we find that

$\lambda_1 = 10/12, \ \lambda_2 = -1/4, \ \lambda_3 = 1.$

$$P^n_{11} = a\left(10/12\right)^n + b\left(-1/4\right)^n + c \cdot 1$$

We have that $P^0_{11} = 1, \ P^1_{11} = 1/2, \ P^2_{11} = 1/2$.

Solving the system of linear equations we find that $a = 9/13, \ b = 4/13, \ c = 0$.

So $P^n_{11} = 9/13\left(10/12\right)^n + 4/13\left(-1/4\right)^n$

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Note that $\lim_{n \to \infty} p_{11}^n = 0$.

ii) $P_{n+1}(A) = P($ it chooses A on the $n+1$ st trial$)$

$P_{n+1}(B) = P($ it chooses B on the $n+1$ st trial$)$

$P_{n+1}(A) = 1 - P_{n+1}(B)$

The only way B can be choosen is if octopus is in state 1 just before the $n+1$ st trial.

So $P_{n+1}(B) = P(B \cap \{X_n = 1\}) = P(B \mid X_n = 1) \cdot P(X_n = 1) = 1/2\, p_{11}^n$

So $P_{n+1}(A) = 1 - 1/2\left(9/13\,(10/12)^n + 4/13\,(-1/4)^n\right)$

Note: $\lambda_1 = 10/12 \sim .833$

$\lambda_2 = -1/4 = -.25$

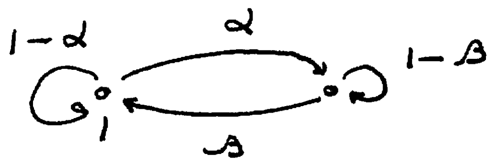
$\lambda_3 = 1$

$9/13 \sim .69$

$4/13 \sim .30$

Part iii) is on the next page.

iii) Suppose that a record of successive choices comes from a 2 state Markov chain.



We interpret $\{X_n = 1\}$ as A is choosen on the nth trial, $\{X_n = 2\}$ as B is choosen on the nth trial.

$$P = \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix} \quad \text{For this MC}$$

$$P^n_{11} = \begin{cases} \dfrac{\beta}{\alpha+\beta} + \dfrac{\alpha}{\alpha+\beta}(1-\alpha-\beta)^n & \text{if } \alpha+\beta > 0 \\ 1 & \text{if } \alpha+\beta = 0 \end{cases}$$

If $\alpha + \beta = 0$. Then $\alpha = \beta = 0$. This would mean that state 2 is terminal. This would not accurately represent our original MC.

Since $\lim_{n\to\infty} P_{n+1}(A) = 1$ in the original MC we need to have $\lim_{n\to\infty} P^n_{11} = 1$ in the new 2 state chain. This is only possible if $\alpha = 0$. This would imply that 1 is the terminal state. So if octopus chooses A on the 1st trial it can never choose B. Again this would not accurately represent our original MC.

We cannot have a 2 state MC.

MATHLAB

```
>> A=[1/2,1/2,0;1/2,1/12,5/12;0,0,1]
A =
    0.5000    0.5000         0
    0.5000    0.0833    0.4167
         0         0    1.0000
>> A^2
ans =
    0.5000    0.2917    0.2083
    0.2917    0.2569    0.4514
         0         0    1.0000
>> eig(A)
ans =
    0.8333_  =  10/12
   -0.2500   =  - 1/4
    1.0000   =  1
>> B=[1,1,1;0.8333,-0.2500,1;(0.8333)^2,(-0.2500)^2,1]
B =
    1.0000    1.0000    1.0000
    0.8333   -0.2500    1.0000
    0.6944    0.0625    1.0000
>> b=[1,1/2,1/2]'
b =
    1.0000
    0.5000
    0.5000
>> inv(B)*b
ans =
    0.6922
    0.3077
    0.0001 = 0.0000
>>
```

The discrepancy in c comes from the fact that 10/12 is .83333_. So c should be identically zero