Chapter 3

Finite Market Model

The binomial model considered in the previous chapter is an example of a finite market model. In that example, we saw that the existence of both a risk neutral probability and a replicating strategy played a key role in justifying the unique arbitrage free price for any European contingent claim. In this chapter, we extend that idea to the pricing of European contingent claims in a general finite market model. We first characterize those finite market models in which there is a risk neutral probability and in which all European contingent claims can be replicated. Indeed, we will prove the fundamental theorem of asset pricing which shows the equivalence of the absence of arbitrage in a finite market model to the existence of a risk neutral probability. It will then be shown that all European contingent claims in a finite market model without arbitrage can be replicated if and only if there is a unique risk neutral probability. Finally, assuming there is such a unique risk neutral probability, we show that there is a unique arbitrage free price for every European contingent claim. In the binomial model, there is a unique risk neutral probability and hence, as shown concretely in the previous chapter, there is a unique arbitrage free price for every European contingent claim.

3.1 Definition of the Finite Market Model

Throughout this chapter we will be working within the framework of the following discrete time, finite state, market model. For short we will call this a finite market model.

Let $(\Omega, \mathcal{F}, P)$ be a probability space where Ω is a finite set of possible outcomes, $\mathcal{F}$ is the σ-algebra consisting of all subsets of Ω and P is a probability measure on $(\Omega, \mathcal{F})$ such that $P(\{\omega\}) > 0$ for all $\omega \in \Omega$. Expectations under P will be written simply as E. Whenever another probability is to be used, this will be explicitly indicated in the notation.

We assume that there are finitely many times $t = 0, 1, \ldots, T-1$ ($T < \infty$) at which trading can occur, and $d+1$ assets, a riskless security called a bond and d risky securities called stocks.

A σ-algebra $\mathcal{F}_t \subset \mathcal{F}$ describes the information available to an investor at time t. It is assumed that $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_T$ and $\mathcal{F}_T = \mathcal{F}$. The collection $\{\mathcal{F}_t, t = 0, 1, \ldots, T\}$ is called a filtration.

The *bond* (asset labelled 0) is assumed to have price process $S^0 = \{S_t^0, t = 0, 1, \ldots, T\}$, where S_t^0 denotes the price of the bond at time t. We assume that for each t, $S_t^0 > 0$ and S_t^0 is deterministic, i.e., $S_t^0 \in \mathcal{F}_0$. For example, if the bond has an interest rate of $r \geq 0$ per unit of time, then $S_t^0 = (1+r)^t$ for all t. The bond is considered to be a "numeraire", i.e., it tells us what a dollar at time 0 is worth (due, amongst other things, to the effects of inflation) at time t.

The d *stocks* are assumed to have price processes $S^1, \ldots, S^d$, where S_t^i is the price of the i^{th} stock at time t. It is assumed that $S_t^i \in \mathcal{F}_t$ for $i = 1, \ldots, d$ and $t = 0, 1, \ldots, T$. Note that since Ω is finite, $S_t = (S_t^0, S_t^1, \ldots, S_t^d)$, $t = 0, 1, \ldots, T$, can take on at most finitely many values. It follows that in the development below, all of the expectations we write will be automatically finite.

A *trading strategy* (in the finite market model) is a collection of $(d+1)$-dimensional vectors indexed by $t = 1, \ldots, T$:

$$H = \{H_t, t = 1, \ldots, T\}, \tag{3.1}$$

where for each $t \in \{1, \ldots, T\}$, $H_t = (H_t^0, H_t^1, \ldots, H_t^d)$ is such that H_t^i is a real-valued $\mathcal{F}_{t-1}$-measurable random variable for $i = 0, 1, \ldots, d$. We regard H_t^i as representing the number of "shares" of asset i to be held over the time interval $(t-1, t]$. In particular, H_t^0 denotes the number of bonds to be held over this interval and H_t^i denotes the number of shares of stock i to be held over the interval, $i = 1, \ldots, d$. A positive value for H_t^i indicates that one buys that number of shares of asset i, at a price of S_{t-1}^i per share, and holds them over the interval $(t-1, t]$. A negative value for H_t^i indicates that asset i will be "sold short". For example, if $H_t^i = -1$, one is effectively borrowing the value S_{t-1}^i of asset i at time $t-1$ with the understanding that the cost to repay this loan at time t is the value of one share of asset i at time t, i.e., S_t^i. We will restrict attention to *self-financing* trading strategies, namely, those trading strategies H such that the investor's initial wealth W_0 is given by

$$W_0 = H_1 \cdot S_0, \tag{3.2}$$

and

$$H_t \cdot S_t = H_{t+1} \cdot S_t, \quad t = 1, \ldots, T-1, \tag{3.3}$$

where $\cdot$ denotes the dot product in $\mathbb{R}^{d+1}$. In this chapter, the term trading strategy will always mean self-financing trading strategy.

The *initial value* of a trading strategy H is $V_0(H) = H_1 \cdot S_0$ and its *value* at time $t \in \{1, \ldots, T\}$ is

$$V_t(H) \equiv H_t \cdot S_t = \sum_{i=0}^{d} H_t^i S_t^i. \tag{3.4}$$

Using the self-financing property we also have that

$$V_t(H) = H_{t+1} \cdot S_t, \quad t = 0, 1, \ldots, T-1. \tag{3.5}$$

The *gains process* associated with a trading strategy H is defined by

$$G_t(H) = V_t(H) - V_0(H), \quad t = 0, 1, \ldots, T. \tag{3.6}$$

Using the equivalent forms for $V_s(H)$ that come from the self-financing property of H, we can rewrite this process as follows for $t = 1, \ldots, T$:

$$\begin{aligned} G_t(H) &= \sum_{s=1}^{t} (V_s(H) - V_{s-1}(H)) \\ &= \sum_{s=1}^{t} (H_s \cdot S_s - H_s \cdot S_{s-1}) \\ &= \sum_{s=1}^{t} H_s \cdot (S_s - S_{s-1}) \\ &= \sum_{s=1}^{t} H_s \cdot \Delta S_s, \end{aligned} \tag{3.7}$$

where $\Delta S_s \equiv S_s - S_{s-1}$. In fact, the last expression is a discrete time stochastic integral (recall that H_s is $\mathcal{F}_{s-1}$ measurable).

An *arbitrage opportunity* (in the finite market model) is a trading strategy H such that

$$V_0(H) = 0, \qquad V_T(H) \geq 0, \qquad E[V_T(H)] > 0.$$

The finite market model is said to be *viable* if it has no arbitrage opportunities.

It will simplify computations to use *discounted asset price processes,* obtained by normalizing so that the value of a dollar at any time t is the same as it is at time 0. This is often called a change of numeraire. For $i = 0, 1, \ldots, d$, we define

$$S_t^{*,i} = \frac{S_t^i}{S_t^0}, \quad \text{for } t = 0, 1, \ldots, T.$$

Note that $S_t^{*,0} \equiv 1$ for all t. Then $S_t^* = (S_t^{*,0}, S_t^{*,1}, \ldots, S_t^{*,d})$ is the value of the vector of discounted asset prices at time t. We will refer to $S^* = \{S_t^*, t =$

$0, 1, \ldots, T\}$ as the (vector) discounted asset price process. The associated discounted value process for a trading strategy H is defined by

$$V_t^*(H) \equiv \frac{V_t(H)}{S_t^0}, \quad t = 0, 1, \ldots, T, \tag{3.8}$$

and using (3.4) and (3.5) we see that

$$V_t^*(H) = H_t \cdot S_t^*, \quad t = 1, \ldots, T, \tag{3.9}$$

and

$$V_t^*(H) = H_{t+1} \cdot S_t^*, \quad t = 0, 1, \ldots, T-1. \tag{3.10}$$

The discounted gains process for H is defined by

$$G_t^*(H) = V_t^*(H) - V_0^*(H), \quad t = 0, 1, \ldots, T, \tag{3.11}$$

and by very similar manipulations to those used in deriving (3.7), this can be reexpressed as $G_0^* = 0$ and

$$G_t^*(H) = \sum_{s=1}^{t} H_s \cdot \Delta S_s^*, \quad t = 1, \ldots, T, \tag{3.12}$$

where $\Delta S_s^* = S_s^* - S_{s-1}^*$. An advantage of this last expression is that it only involves the risky assets, since $\Delta S_s^{*,0} = 0$ for $s = 1, \ldots, T$.

3.2 Fundamental Theorem of Asset Pricing

The following definitions will be needed to state the fundamental theorem of asset pricing, which characterizes viable finite market models.

Definition 3.2.1 *Two probability measures Q and Q' on $(\Omega, \mathcal{F})$ are equivalent (or mutually absolutely continuous) if*

$$Q(A) = 0 \quad \text{is equivalent to} \quad Q'(A) = 0 \quad \text{for all } A \in \mathcal{F}. \tag{3.13}$$

In the finite market model, P gives positive probability to every $\omega \in \Omega$, and so for a probability measure P^* on $(\Omega, \mathcal{F})$, P is equivalent to P^* if and only if $P^*(\{\omega\}) > 0$ for all $\omega \in \Omega$.

Definition 3.2.2 *An equivalent martingale measure (abbreviated as EMM) is a probability measure P^* defined on $(\Omega, \mathcal{F})$ such that P^* is equivalent to P and S^* is a martingale under P^* (relative to the filtration $\{\mathcal{F}_t, t = 0, 1, \ldots, T\}$), i.e., for each $t \in \{1, \ldots, T\}$,*

$$E^{P^*}[S_t^* \mid \mathcal{F}_{t-1}] = S_{t-1}^*, \tag{3.14}$$

where E^{P^*} denotes expectation under P^* and the above equality is to be interpreted componentwise.

We note for future use that (3.14) is equivalent to

$$E^{P^*}[\Delta S_t^* \mid \mathcal{F}_{t-1}] = 0. \tag{3.15}$$

Remark. An equivalent martingale measure is sometimes also called a *risk neutral probability.* We will use the terms interchangeably.

Theorem 3.2.3 *(Fundamental Theorem of Asset Pricing) The finite market model is viable if and only if there exists an equivalent martingale measure P^*.*

PROOF. We first prove the "if" part of the theorem. Suppose there exists an equivalent martingale measure P^*. For a proof by contradiction, suppose that H is an arbitrage opportunity, that is, H is a trading strategy with initial value $V_0(H) = 0$, final value $V_T(H) \geq 0$, and $E[V_T(H)] > 0$. It follows that the discounted values satisfy $V_0^*(H) = 0$, $V_T^*(H) \geq 0$, and since P^* is equivalent to P, $E^{P^*}[V_T^*(H)] > 0$. Then, by (3.11) and (3.12) we have

$$V_T^*(H) = V_0^*(H) + G_T^*(H) = 0 + \sum_{t=1}^{T} H_t \cdot \Delta S_t^*. \tag{3.16}$$

On taking expectations we obtain

$$\begin{aligned}
E^{P^*}[V_T^*(H)] &= E^{P^*}\left[\sum_{t=1}^{T} H_t \cdot \Delta S_t^*\right] \\
&= \sum_{t=1}^{T} E^{P^*}\left[E^{P^*}[H_t \cdot \Delta S_t^* \mid \mathcal{F}_{t-1}]\right] \\
&= \sum_{t=1}^{T} E^{P^*}\left[H_t \cdot E^{P^*}[\Delta S_t^* \mid \mathcal{F}_{t-1}]\right] \\
&= 0,
\end{aligned}$$

since $H_t \in \mathcal{F}_{t-1}$ and by the martingale property (3.15). But this contradicts $E^{P^*}[V_T^*(H)] > 0$, and so there cannot be an arbitrage opportunity in the finite market model, and hence the model is viable.

We now turn to proving the "only if" part of the theorem. For this, suppose that the finite market model is viable. Since Ω is a finite set, for any random variable U defined on $(\Omega, \mathcal{F})$, by enumerating Ω as $\{\omega_1, \ldots, \omega_n\}$, we may view U as $(U(\omega_1), \ldots, U(\omega_n)) \in \mathbb{R}^n$. Thus, there is a one-to-one correspondence between points in $\mathbb{R}^n$ and (real-valued) random variables defined on Ω. Adopting this point of view for the terminal discounted gain random variables $G_T^*(H)$, we define

$$L = \{G_T^*(H) : H \text{ is a (self-financing) trading strategy such that } V_0(H) = 0\}.$$

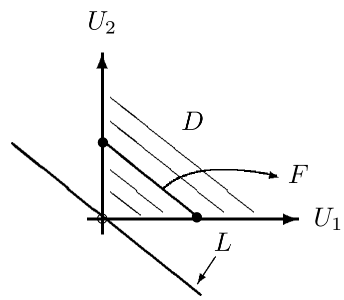


Figure 3.1: An example of the situation for $n = 2$

Note that L is a linear space, since $G_T^*(H)$ is linear in H and any linear combination of (self-financing) trading strategies with initial values of zero is again a trading strategy with the same initial value. Also, L is non-empty because the origin is contained in L. Let

$$D = \{U \in \mathbb{R}^n : U_i \geq 0 \text{ for } i = 1, \ldots, n \text{ and } U_j > 0 \text{ for some } j\}. \tag{3.17}$$

Thus, D is the positive orthant in $\mathbb{R}^n$ with the origin removed.

Since the market is assumed to be viable, $L \cap D = \emptyset$. For otherwise there would be a trading strategy H with $V_0(H) = 0$, $V_T^*(H) = G_T^*(H) \geq 0$ and $V_T^*(H)(\omega_i) > 0$ for at least one i, which would represent an arbitrage opportunity. Let

$$F = \left\{ U \in D : \sum_{i=1}^{n} U_i = 1 \right\}. \tag{3.18}$$

Then F is a convex, compact, non-empty subset of $\mathbb{R}^n$ and $L \cap F = \emptyset$.

By applying the separating Hyperplane Theorem 3.5.1, we see that there is a vector $W \in \mathbb{R}^n \setminus \{0\}$ such that the hyperplane $N = \{U \in \mathbb{R}^n : U \cdot W = 0\}$ contains L and $W \cdot U > 0$ for all $U \in F$. By setting $U_i = 1$ if $i = j$ and $U_i = 0$ if $i \neq j$, we see that $W_j > 0$ for each $j \in \{1, \ldots, n\}$. Define

$$P^*(\{\omega_i\}) = \frac{W_i}{\sum_{j=1}^n W_j}, \quad i = 1, \ldots, n. \tag{3.19}$$

Then P^* is a probability measure on $(\Omega, \mathcal{F})$ and it is equivalent to P. Moreover, for any trading strategy H such that $V_0(H) = 0$, we have

$$E^{P^*}[G_T^*(H)] = \sum_{i=1}^{n} G_T^*(H)(\omega_i) \frac{W_i}{\sum_{j=1}^n W_j} \tag{3.20}$$

$$= \frac{G_T^*(H) \cdot W}{\sum_{j=1}^n W_j} \tag{3.21}$$

$$= 0, \tag{3.22}$$

where the last line follows from the fact that W is perpendicular to N which contains L.

Note that $G_T^*(H)$ only involves $(H^1, \ldots, H^d)$. From Lemma 3.2.4, proved below, given $\hat{H}^1, \ldots, \hat{H}^d$, where for $i = 1, \ldots, d$, $\hat{H}^i = \{\hat{H}_t^i, t = 1, \ldots, T\}$ and $\hat{H}_t^i$ is a real-valued, $\mathcal{F}_{t-1}$-measurable random variable for each t, there is a unique ordered set of T real-valued random variables $H^0 = \{H_t^0, t = 1, \ldots, T\}$ such that $H \equiv \{(H_t^0, \hat{H}_t^1, \ldots, \hat{H}_t^d), t = 1, \ldots, T\}$ is a (self-financing) trading strategy with an initial value of zero. Upon substituting this in (3.20) and writing out the expression (cf. (3.12)) for $G_T^*(H)$, we see that

$$\begin{aligned} 0 = E^{P^*}\left[G_T^*(H)\right] &= E^{P^*}\left[\sum_{t=1}^T H_t \cdot \Delta S_t^*\right] && (3.23) \\ &= E^{P^*}\left[\sum_{t=1}^T \sum_{i=1}^d \hat{H}_t^i \Delta S_t^{*,i}\right]. && (3.24) \end{aligned}$$

For each fixed $i \in \{1, \ldots, d\}$, if we set $\hat{H}_t^j = 0$ for all t and $j \neq i$, we obtain

$$0 = E^{P^*}\left[\sum_{t=1}^T H_t^i \Delta S_t^{*,i}\right], \tag{3.25}$$

for each $\hat{H}^i = \{\hat{H}_t^i, t = 1, \ldots, T\}$ such that $\hat{H}_t^i$ is a real-valued $\mathcal{F}_{t-1}$-measurable random variable for each t. It then follows from Lemma 3.2.5, proved below, that for $i = 1, \ldots, d$, $S^{*,i}$ is a martingale under P^*. Hence, P^* is an equivalent martingale measure. □

The next two lemmas were used in the above proof of the Fundamental Theorem of Asset Pricing. The first lemma shows that given (non-anticipating) holdings in the risky assets and an initial wealth, there is a unique sequence of holdings in the riskless asset that makes the associated trading strategy self-financing.

Lemma 3.2.4 *For $i = 1, \ldots, d$, let $\hat{H}^i = \{\hat{H}_t^i, t = 1, \ldots, T\}$ where $\hat{H}_t^i$ is a real-valued, $\mathcal{F}_{t-1}$-measurable random variable for $t = 1, \ldots, T$. For each real-valued $\mathcal{F}_0$-measurable random variable V_0, there exists a unique ordered set of T real-valued random variables $H^0 = \{H_t^0, t = 1, \ldots, T\}$ such that $H \equiv \{(H_t^0, \hat{H}_t^1, \ldots, \hat{H}_t^d), t = 1, \ldots, T\}$ is a (self-financing) trading strategy with an initial value of V_0.*

Proof. Fix $V_0 \in \mathcal{F}_0$. For H to be self-financing at time zero, we must have (cf. (3.2)):

$$H_1 \cdot S_0 = V_0 \tag{3.26}$$

and since $\hat{H}_1^1, \ldots, \hat{H}_1^d$ are given, this will be satisfied if and only if

$$H_1^0 = (S_0^0)^{-1}\left(V_0 - \sum_{i=1}^{d} \hat{H}_1^i S_0^i\right). \tag{3.27}$$

Note that this $H_1^0 \in \mathcal{F}_0$. Thus, H_1^0 is uniquely determined. For an induction, suppose that for some $1 \le s \le T-1$, H_t^0, $t = 1, \ldots, s$, have been determined uniquely such that $H_t^0 \in \mathcal{F}_{t-1}$ for each $t = 1, \ldots, s$, (3.26) holds, and

$$H_t \cdot S_t = H_{t+1} \cdot S_t, \quad t = 1, \ldots, s-1. \tag{3.28}$$

Then, the self-financing property (3.28) holds for $t = s$ if and only if we have

$$H_{s+1}^0 = (S_s^0)^{-1}\left(H_s \cdot S_s - \sum_{i=1}^{d} \hat{H}_{s+1}^i S_s^i\right). \tag{3.29}$$

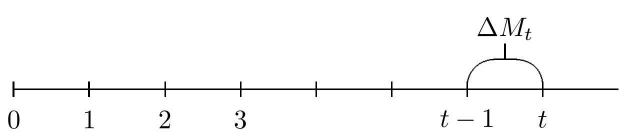
Note that this expression for H_{s+1}^0 is $\mathcal{F}_s$-measurable. This establishes the induction step and it follows that there is a unique H^0 that makes H a (self-financing) trading strategy with initial value V_0. □

Lemma 3.2.5 *Let $M = \{M_t, t = 0, 1, \ldots, T\}$ be a real-valued process such that $M_t \in \mathcal{F}_t$ for each t. Then, M is a martingale if and only if*

$$E\left[\sum_{t=1}^{T} \eta_t \Delta M_t\right] = 0 \tag{3.30}$$

for all $\eta = \{\eta_t, t = 1, \ldots, T\}$ such that η_t is a real-valued $\mathcal{F}_{t-1}$-measurable random variable for $t = 1, \ldots, T$. Here, $\Delta M_t = M_t - M_{t-1}$ for $t = 1, \ldots, T$.

Remark. The sum $\sum_{t=1}^{T} \eta_t \Delta M_t$ is actually a discrete stochastic integral. If one extends η to a continuous time process by making it constant on $(t-1, t]$ with a value of η_t there, and one extends M to be constant on $[t-1, t)$ with a value of M_{t-1} there, then the sum is the same as the stochastic integral $\int_{[0,T]} \eta_t dM_t$.



Proof. Suppose M is a martingale. Let $\eta = \{\eta_t, t = 1, \ldots, T\}$ where η_t is a real-valued $\mathcal{F}_{t-1}$-measurable random variable for each t. Then, since $\eta_t \in \mathcal{F}_{t-1}$, we have

$$E\left[\sum_{t=1}^{T} \eta_t \Delta M_t\right] = \sum_{t=1}^{T} E\left[\eta_t E\left[\Delta M_t \mid \mathcal{F}_{t-1}\right]\right]. \tag{3.31}$$

Now, because M is a martingale we have $E[\Delta M_t \mid \mathcal{F}_{t-1}] = 0$, for $t = 1, \ldots, T$, and it follows that (3.30) holds.

Conversely, suppose that (3.30) holds for all $\eta = \{\eta_t, t = 1, \ldots, T\}$ where η_t is a real-valued $\mathcal{F}_{t-1}$-measurable random variable for each t. For fixed $s \in \{1, \ldots, T\}$ and $A \in \mathcal{F}_{s-1}$, let

$$\eta_t = \begin{cases} 0 & \text{for } t \neq s, \\ 1_A & \text{for } t = s. \end{cases}$$

Then $\eta_t \in \mathcal{F}_{t-1}$ for each t. Upon substituting this into (3.30), we obtain

$$E[1_A \Delta M_s] = 0. \tag{3.32}$$

Since $A \in \mathcal{F}_{s-1}$ was arbitrary, it follows that

$$E[M_s \mid \mathcal{F}_{s-1}] = M_{s-1}, \tag{3.33}$$

and then since s was arbitrary, it follows that M is a martingale. □

3.3 European Contingent Claims

A *European contingent claim* is represented by a $\mathcal{F}_T$-measurable random variable X. The value (or payoff) of the contingent claim at the exercise time T is X. For example, a European call option with strike price K and expiration date T that is based on the risky asset with price process S^1 is represented by $X = (S_T^1 - K)^+$. On the other hand, a look-back option usually depends on the recent history of a risky asset (see the Exercises for an example of such).

We will frequently refer to the random variable X, that represents a European contingent claim, as the European contingent claim (or ECC for short). For a European contingent claim X, we let $X^* = X/S_T^0$, the discounted value of X.

A *replicating (or hedging) strategy* for a European contingent claim X is a trading strategy H such that $V_T(H) = X$. If there exists such a replicating strategy, the European contingent claim is said to be *attainable.*

The finite market model is said to be *complete* if all European contingent claims are attainable.

Theorem 3.3.1 *Suppose that the finite market model is viable and X is a replicable European contingent claim. Then the value process $\{V_t(H), t = 0, 1, \ldots, T\}$ is the same for all replicating strategies H for X. Indeed, for any replicating strategy H,*

$$V_t^*(H) = E^{P^*}[X^* \mid \mathcal{F}_t], \quad t = 0, 1, \ldots, T, \tag{3.34}$$

for any equivalent martingale measure P^*, and the right member of (3.34) has the same value for all such P^*.

PROOF. By the Fundamental Theorem of Asset Pricing, there is at least one equivalent martingale measure. Let P^* be such a measure. Let H be a replicating strategy for X. Then, for $t = 0, 1, \ldots, T$, by the martingale property of S^* under P^*,

$$E^{P^*}\left[\sum_{s=t+1}^{T} H_s \cdot \Delta S_s^* \,\middle|\, \mathcal{F}_t\right] = \sum_{s=t+1}^{T} E^{P^*}\left[H_s \cdot E^{P^*}\left[\Delta S_s^* \mid \mathcal{F}_{s-1}\right] \,\middle|\, \mathcal{F}_t\right] = 0, \tag{3.35}$$

and it follows that (cf. (3.11)–(3.12)):

$$\begin{aligned} V_t^*(H) &= V_0(H) + \sum_{s=1}^{t} H_s \cdot \Delta S_s^* && (3.36)\\ &= V_0(H) + \sum_{s=1}^{t} H_s \cdot \Delta S_s^* + E^{P^*}\left[\sum_{s=t+1}^{T} H_s \cdot \Delta S_s^* \,\middle|\, \mathcal{F}_t\right] && (3.37)\\ &= E^{P^*}\left[V_T^*(H) \mid \mathcal{F}_t\right] && (3.38)\\ &= E^{P^*}\left[X^* \mid \mathcal{F}_t\right], && (3.39) \end{aligned}$$

where $X^* = X/S_T^0$. Since the last line above does not depend upon H, it follows that $V_t^*(H)$ and hence $V_t(H)$ does not depend upon the particular choice of replicating strategy H. Furthermore, since the right member of (3.36) does not depend upon the particular choice of an EMM P^*, it follows that the quantity in (3.39) also has this property. □

Theorem 3.3.2 *A viable finite market model is complete if and only if it admits a unique equivalent martingale measure.*

PROOF. Suppose the market is viable and complete. Let Q and $\tilde{Q}$ be equivalent martingale measures. Fix $A \in \mathcal{F}_T$ and let $X = 1_A$. Suppose that H is a replicating strategy for X. Then, by Theorem 3.3.1

$$E^Q[X^*] = E^{\tilde{Q}}[X^*]. \tag{3.40}$$

Multiplying both sides by the deterministic quantity S_T^0 yields

$$E^Q[X] = E^{\tilde{Q}}[X] \tag{3.41}$$

and so

$$Q(A) = \tilde{Q}(A). \tag{3.42}$$

Hence, $Q = \tilde{Q}$, since $A \in \mathcal{F}_T = \mathcal{F}$ was arbitrary.

Conversely, suppose the market is viable but *not* complete. Then we will show that there is more than one equivalent martingale measure. Since the market is not complete, there exists a European contingent claim X that is not attainable. Let $\mathcal{P}$ denote the set of $\hat{H} = (\hat{H}^1, \ldots, \hat{H}^d)$ satisfying $\hat{H}^i = \{\hat{H}^i_t, t = 1, \ldots, T\}$ where $\hat{H}^i_t$ is a real-valued $\mathcal{F}_{t-1}$-measurable random variable for $i = 1, \ldots, d$, $t = 1, \ldots, T$. We claim that there is no pair $(\hat{H}, c)$ such that $\hat{H} \in \mathcal{P}$, $c \in \mathbb{R}$ and

$$c + \sum_{t=1}^{T} \hat{H}_t \cdot \Delta \hat{S}^*_t = X^*, \tag{3.43}$$

where $\hat{S}^* = (S^{*,1}, \ldots, S^{*,d})$. For, if there were such a pair $(c, \hat{H})$, then by Lemma 3.2.4, $\hat{H}$ could be extended to a trading strategy $H = (H^0, \hat{H}^1, \ldots, \hat{H}^d)$ with initial value c, and then H would be a replicating strategy for X.

Now, adopting the same device as in the proof of Theorem 3.2.3 of viewing random variables as vectors in $\mathbb{R}^n$, let

$$L = \left\{ c + \sum_{t=1}^{T} \hat{H}_t \cdot \Delta \hat{S}^*_t : \hat{H} \in \mathcal{P},\ c \in \mathbb{R} \right\}. \tag{3.44}$$

Then, L is a linear subspace of $\mathbb{R}^n$ and $X^* \notin L$. It follows that L is a strict subspace of $\mathbb{R}^n$ and there is a non-zero vector $U \in \mathbb{R}^n$ such that $U \in L^\perp$. Then,

$$\sum_{\omega \in \Omega} U(\omega) W(\omega) = 0 \qquad \text{for all} \;\; W \in L. \tag{3.45}$$

Since the finite market model is viable, there is at least one equivalent martingale measure P^*. Then $P^*(\{\omega\}) > 0$ for all $\omega \in \Omega$, and on setting

$$\tilde{U}(\omega) = \frac{U(\omega)}{P^*(\omega)} \quad \text{for all } \omega \in \Omega, \tag{3.46}$$

we may rewrite (3.45) as

$$E^{P^*}[\tilde{U} W] = 0 \quad \text{for all } W \in L. \tag{3.47}$$

Now, define

$$P^{**}(\{\omega\}) = \left(1 + \frac{\tilde{U}(\omega)}{2 \|\tilde{U}\|_\infty} \right) P^*(\{\omega\}) \qquad \text{for each } \omega \in \Omega, \tag{3.48}$$

where $\|\tilde{U}\|_\infty = \max_{\omega \in \Omega} |\tilde{U}(\omega)|$. Since $\tilde{U} \neq 0$, $P^{**} \neq P^*$. Moreover, $P^{**}(\{\omega\}) > 0$ for each $\omega \in \Omega$. To check that P^{**} is a probability measure, note that

$$\begin{aligned} P^{**}(\Omega) &= P^*(\Omega) + \sum_{\omega \in \Omega} \frac{\tilde{U}(\omega)}{2\,\|\tilde{U}\|_\infty} P^*(\{\omega\}) && (3.49) \\ &= 1 + \frac{1}{2\,\|\tilde{U}\|_\infty} E^{P^*}\left[\tilde{U}\right] && (3.50) \\ &= 1, && (3.51) \end{aligned}$$

where we used (3.47) with $W = 1 \in L$ to obtain the last line. Thus, P^{**} is a probability measure that is equivalent to P^*. We finally need to check that S^* is a martingale under P^{**}. For any $\hat{H} \in \mathcal{P}$,

$$E^{P^{**}}\left[\sum_{t=1}^{T} \hat{H}_t \cdot \Delta \hat{S}_t^*\right] = E^{P^*}\left[\sum_{t=1}^{T} \hat{H}_t \cdot \Delta \hat{S}_t^*\right] + \frac{1}{2\,\|\tilde{U}\|_\infty} E^{P^*}\left[\tilde{U} \sum_{t=1}^{T} \hat{H}_t \cdot \Delta \hat{S}_t^*\right]. \tag{3.52}$$

The first term in the right member above is zero, by Lemma 3.2.5, since S^* is a martingale under P^*. The second term there is zero by (3.47), since $W = \sum_{t=1}^{T} \hat{H}_t \cdot \Delta \hat{S}_t^* \in L$. On applying Lemma 3.2.5 again, it follows that $S^{*,i}$ is a P^{**}-martingale for $i = 1, \ldots, d$ and since this is trivially so for $i = 0$, it follows that S^* is a P^{**} martingale and hence P^{**} is an equivalent martingale measure that is different from P^*. □

The following is a form of the martingale representation theorem in a finite market model context.

Theorem 3.3.3 *Suppose the finite market model is viable and P^* is an equivalent martingale measure. Then, the model is complete if and only if each real-valued martingale $M = \{M_t, t = 0, 1, \ldots, T\}$ under P^* has a representation of the form*

$$M_t = M_0 + \sum_{s=1}^{t} \hat{H}_s \cdot \Delta \hat{S}_s^*, \qquad t = 0, 1, \ldots, T, \tag{3.53}$$

for some $\hat{H} = (\hat{H}^1, \ldots, \hat{H}^d)$ where for each $i = 1, \ldots, d$, $\hat{H}^i = \{\hat{H}_t^i, t = 1, \ldots, T\}$ and $\hat{H}_t^i$ is a real-valued $\mathcal{F}_{t-1}$-measurable random variable for each t. (Here $\hat{S}^* = (S^{*,1}, S^{*,2}, \ldots, S^{*,d})$.)

Proof. Suppose the model is complete and let $M = \{M_t, t = 0, 1, \ldots, T\}$ be a martingale under P^*. Then, $X = M_T S_T^0$ is a European contingent claim. Since the model is complete, there exists a replicating strategy H for X. Then, $V_T^*(H) = X^* = M_T$ and by Theorem 3.3.1 we have for $t = 0, 1, \ldots, T-1$,

$$V_t^*(H) = E^{P^*}[X^* \mid \mathcal{F}_t] = E^{P^*}[M_T \mid \mathcal{F}_t]. \tag{3.54}$$

Since M is a P^*-martingale, the last member above is equal to M_t P^*-a.s. and so it follows upon using (3.11)–(3.12) that for $t = 0, 1, \ldots, T$,

$$\begin{aligned} M_t = V_t^*(H) &= V_0 + G_t^*(H) && (3.55)\\ &= V_0 + \sum_{s=1}^{t} H_s \cdot \Delta S_s^* && (3.56)\\ &= M_0 + \sum_{s=1}^{t} \hat{H}_s \cdot \Delta \hat{S}_s^* && (3.57) \end{aligned}$$

where $\hat{H} = \{(H_t^1, \ldots, H_t^d), t = 1, \ldots, T\}$ and $H_t^i \in \mathcal{F}_{t-1}$ for $t = 1, \ldots, T$, $i = 1, \ldots, d$. Here we have used the fact that $G_t^*(H)$ only involves the holdings in the risky assets.

Conversely, suppose the representation property holds. To show that the model is complete, consider a European contingent claim X. Define

$$M_t = E^{P^*}[X^* \mid \mathcal{F}_t], \quad t = 0, 1, \ldots, T. \tag{3.58}$$

Then M is a martingale under P^*. Let $\hat{H}$ be as in the representation (3.53). By Lemma 3.2.4, $\hat{H}$ can be extended to a (self-financing) trading strategy $H = (H^0, \hat{H}^1, \ldots \hat{H}^d)$ with initial value M_0. Then, for $t = 0, 1, \ldots, T$,

$$\begin{aligned} V_t^*(H) &= V_0(H) + G_t^*(H) && (3.59) \\ &= M_0 + \sum_{s=1}^{t} \hat{H}_s \cdot \Delta \hat{S}_s^* && (3.60) \\ &= M_t. && (3.61) \end{aligned}$$

Hence, $V_T^*(H) = M_T = X^*$ and it follows that $V_T(H) = X$. Thus, H is a replicating strategy for X. Since X was an arbitrary European contingent claim, it follows that the market is complete. □

3.4 Arbitrage-Free Price Process

In this section, we assume that the finite market model is viable and complete. Let P^* be the unique equivalent martingale measure. Consider a European contingent claim with value X at time T. To determine the arbitrage free price process for the European contingent claim, we consider a market that allows trading in the stock, bond and European contingent claim at each time $t = 0, 1, \ldots, T-1$ (this is in contrast to section 2.3 where we only allowed trading in the European contingent claim at time zero).

Let $\{C_t, t = 0, 1, \ldots, T\}$ be an adapted process, where C_t represents the price of the European contingent claim at time $t = 0, 1, \ldots, T-1$ and $C_T = X$. A *trading strategy* in stocks, bond and the European contingent claim is a collection $J = \{(H_t, \gamma_t), t = 1, \ldots, T\}$ where for each t, $H_t = (H_t^0, H_t^1, \ldots, H_t^d)$ is a $(d+1)$-dimensional $\mathcal{F}_{t-1}$-measurable random vector such that for $i = 0, 1, \ldots, d$, H_t^i represents the number of "shares" of asset i held over the time interval $(t-1, t]$, and γ_t is a real-valued $\mathcal{F}_{t-1}$-measurable random variable representing the number of European contingent claims held over the time interval $(t-1, t]$. This trading strategy must be self-financing, i.e., its initial value is

$$V_0(J) = H_1 \cdot S_0 + \gamma_1 C_0,$$

and at each time $t = 1, \dots, T-1$,

$$H_t \cdot S_t + \gamma_t C_t = H_{t+1} \cdot S_t + \gamma_{t+1} C_t.$$

The value of the stocks-bond-contingent claim portfolio at time T is

$$V_T(J) = H_T \cdot S_T + \gamma_T X.$$

An *arbitrage opportunity* in the stocks-bond-contingent claim market is a (self-financing) trading strategy J such that $V_0(J) = 0$, $V_T(J) \geq 0$ and $E[V_T(J)] > 0$.

Theorem 3.4.1 *Suppose the finite market model is viable and complete and P^* is the unique equivalent martingale measure. Then for any European contingent claim X,*

$$\{S_t^0 E^{P^*}[X^* \mid \mathcal{F}_t], t = 0, 1, \dots, T\}$$

is the (unique) arbitrage free price process for the European contingent claim, where $X^* = X/S_T^0$ is the discounted value of X at time T.

Proof. Let H be a replicating strategy for X. Then by Theorem 3.3.1,

$$V_t(H) = S_t^0 E^{P^*}[X^* \mid \mathcal{F}_t], \quad t = 0, 1, \dots, T.$$

Let $\{C_t, t = 0, 1, \dots, T\}$ be the price process for the European contingent claim, where $C_T = X$.

We first show that if $P(C_s \neq V_s(H)) > 0$ for some s, then there is an arbitrage opportunity. Note $C_T = V_T(H) = X$. Suppose there is $s \in \{0, 1, \dots, T-1\}$ such that $P(C_s > V_s(H)) > 0$. Let $A = \{\omega : C_s(\omega) > V_s(H)(\omega)\}$. Then $A \in \mathcal{F}_s$. An investor could act as follows to achieve an arbitrage. The investor invests nothing in stocks, bond or contingent claim up to time s. If $C_s \leq V_s(H)$, the investor continues to invest nothing from time s to T. If $C_s > V_s(H)$, at time s, the investor sells (short) one European contingent claim, invests $V_s(H)$ of the proceeds in the market from time s onwards according to the strategy $(H_{s+1}, \dots, H_T)$, and puts the remainder, $C_s - V_s(H)$, in the bond from time s to T. This strategy may be formally written as

$$J_t = \begin{cases} 0 & \text{for } t \leq s \\ \left(H_t^0 + \frac{C_s - V_s(H)}{S_s^0}, H_t^1, \dots, H_t^d, -1\right) 1_A & \text{for } t > s. \end{cases}$$

It is readily verified to be self-financing. In particular, $V_t(J) = 0$ for $t \leq s$, and

$$1_A(H_{s+1} \cdot S_s + C_s - V_s(H) - C_s) = 0.$$

Now,

$$\begin{aligned} V_T(J) &= 1_A\left(H_T \cdot S_T + (C_s - V_s(H))\frac{S_T^0}{S_s^0} - X\right) \\ &= 1_A\left((C_s - V_s(H))\frac{S_T^0}{S_s^0}\right), \end{aligned}$$

since $V_T(H) = X$, by the replicating property of H. Thus, $V_T(J) \geq 0$ and $P(V_T(J) > 0) = P(A) > 0$, and J is an arbitrage.

Similarly, if $s \in \{0, 1, \ldots, T-1\}$ such that $P(C_s < V_s(H)) > 0$ and $B = \{C_s < V_s(H)\}$, and we let

$$J_t = \begin{cases} 0 & \text{for } t \leq s \\ \Big(-H_t^0 + \frac{V_s(H)-C_s}{S_s^0}, -H_t^1, \ldots, -H_t^d, 1\Big) 1_B & \text{for } t > s, \end{cases}$$

for $t = 1, \ldots, T$, then J represents an arbitrage. Thus we have shown that if $P(C_s \neq V_s(H)$ for some $s) > 0$, there is an arbitrage opportunity.

It remains to show that if $C_s = V_s(H)$ for all s, there is no arbitrage opportunity in the stocks-bond-contingent claim market. For a contradiction, suppose $C_s = V_s(H)$ for all s and that $J = \{(\tilde{H}_t, \tilde{\gamma}_t), t = 1, \ldots, T\}$ is an arbitrage opportunity. Then $V_0(J) = 0$, $V_T(J) \geq 0$ and $E[V_T(J)] > 0$. Let $V_t^*(J) = V_t(J)/S_t^0, t = 0, 1, \ldots, T$. Then

$$\begin{aligned} E^{P^*}[V_T^*(J)] &= E^{P^*}[\tilde{H}_T \cdot S_T^* + \tilde{\gamma}_T V_T^*(H)] \\ &= E^{P^*}[\tilde{H}_T \cdot E^{P^*}[S_T^* \mid \mathcal{F}_{T-1}] + \tilde{\gamma}_T E^{P^*}[V_T^*(H) \mid \mathcal{F}_{T-1}]] \\ &= E^{P^*}[\tilde{H}_T \cdot S_{T-1}^* + \tilde{\gamma}_T V_{T-1}^*(H)] \end{aligned}$$

where we have used the fact that S^* and $V^*(H)$ are martingales under P^* (cf. (3.32)). Using the self-financing property of J, we recognize the above as $E^{P^*}[V_{T-1}^*(J)]$. One can repeat a similar argument $T-1$ more times to obtain,

$$E^{P^*}[V_T^*(J)] = E^{P^*}[V_0^*(J)].$$

Now, $V_0^*(J) = V_0(J) = 0$ and $V_T^*(J) \geq 0$, so it follows that $P^*(V_T^*(J) = 0) = 1$ and since P^* is equivalent to P, $P(V_T^*(J) = 0) = 1$ and hence $P(V_T(J) = 0) = 1$, which contradicts the assumption that J is an arbitrage.

Thus, $C_t = V_t(H) = S_t^0 E^{P^*}[X^* \mid \mathcal{F}_t]$, $t = 0, 1, \ldots, T$, defines the arbitrage free price process. □

Remark: Examination of the above proof shows that the conclusion of the theorem still holds if the finite market model is viable and X is simply replicable, i.e., one does not need to assume completeness of the market model.

3.5 Separating Hyperplane Theorem

Theorem 3.5.1 *(Separating Hyperplane Theorem) Let F be a compact, convex, non-empty subset of $\mathbb{R}^n$. Let L be a non-empty linear subspace of $\mathbb{R}^n$. Suppose $F \cap L = \emptyset$. Then there exists a hyperplane*

$$N = \{u \in \mathbb{R}^n : w \cdot u = 0\} \text{ for some } w \in \mathbb{R}^n \setminus \{0\}$$

such that $L \subset N$ and $w \cdot u > 0$ for all $u \in F$.

PROOF. Define $G = F - L = \{u \in \mathbb{R}^n : u = f - \ell \text{ for some } f \in F, \ell \in L\}$. Then G is convex, closed and non-empty. The convexity follows easily from that for F and L. To see that G is closed, consider a sequence $\{u_m = f_m - \ell_m\}_{m=1}^{\infty}$ in G where $f_m \in F$, $\ell_m \in L$ for all m. Suppose $u_m \to u \in \mathbb{R}^n$ as $m \to \infty$. Since F is compact, there is a subsequence $\{f_{m_r}\}_{r=1}^{\infty}$ converging to some $f \in F$. Then $\ell_{m_r} = -u_{m_r} + f_{m_r}$ converges to $-u + f$. Since L is closed we must have $-u + f \in L$. Hence $u = f - (-u + f) \in G$. Clearly G is non-empty, since F and L are both non-empty. Note that G does not contain the origin (otherwise F would intersect L non-trivially).

Let $B = B(0, r)$ denote the closed ball centered at the origin of radius $r > 0$. Choose $r > 0$ such that $B \cap G \neq \emptyset$. Then $B \cap G$ is closed, bounded, non-empty and hence compact. So the continuous function $g(u) = \|u\|$ attains its infimum on $B \cap G$ at some $w \in B \cap G$, where $\|u\| = (u \cdot u)^{\frac{1}{2}}$ denotes the Euclidean norm of u.

Now $\|u\| > r$ for $u \in G \setminus B$ and so combining this with the above we have $\|u\| \geq \|w\|$ for all $u \in G$. Then for any $\lambda \in (0, 1)$ and $u \in G$, $\lambda u + (1 - \lambda)w \in G$ by the convexity of G and so

$$\|\lambda u + (1 - \lambda)w\|^2 \geq \|w\|^2 \text{ for all } u \in G, \lambda \in (0, 1).$$

Expanding and dividing through by λ yields:

$$2(1 - \lambda)u \cdot w - 2w \cdot w + \lambda(u \cdot u + w \cdot w) \geq 0.$$

Letting $\lambda \to 0$, we obtain

$$u \cdot w \geq w \cdot w \quad \text{for all } u \in G.$$

Then, $(f - \ell) \cdot w \geq w \cdot w$ for all $f \in F$, $\ell \in L$, which implies

$$f \cdot w \geq \ell \cdot w + w \cdot w \quad \text{for all } f \in F, \ell \in L.$$

Fix $f \in F$. Then

$$\ell \cdot w \leq f \cdot w - w \cdot w \quad \text{for all } \ell \in L.$$

But L is a linear space, so the above holds with $\gamma\ell$ in place of ℓ for all $\gamma \in \mathbb{R}$. The only way this can be true is if $\ell \cdot w = 0$ for all $\ell \in L$. Hence, $f \cdot w \geq w \cdot w > 0$ for all $f \in F$.

Let $N = \{u \in \mathbb{R}^n : u \cdot w = 0\}$. Then from the above, $L \subset N$ and $f \cdot w > 0$ for all $f \in F$. □

3.6 Exercises

1. Consider the multi-period CRR binomial model introduced in Chapter 2. Assuming $u > d > 0$, verify that this model is viable if and only if $d < 1 + r < u$. In this case, verify that the model is complete.

2. Let $T = 2$, $\Omega = \{\omega_1, \ldots, \omega_4\}$, $P(\{\omega_i\}) > 0$ for $i = 1, \ldots, 4$, $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_1 = \{\{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}, \emptyset, \Omega\}$, and $\mathcal{F}_2$ be the collection of all subsets of Ω. Consider a riskless asset with price process $S^0 = \{S_t^0, t = 0, 1, 2\}$ where $S_t^0 = 1$ for all t, and a risky asset with price process $S^1 = \{S_t^1, t = 0, 1, 2\}$ such that

$$S_0^1(\omega_1) = 5, \quad S_1^1(\omega_1) = 8, \quad S_2^1(\omega_1) = 9 \tag{3.62}$$
$$S_0^1(\omega_2) = 5, \quad S_1^1(\omega_2) = 8, \quad S_2^1(\omega_2) = 6 \tag{3.63}$$
$$S_0^1(\omega_3) = 5, \quad S_1^1(\omega_3) = 4, \quad S_2^1(\omega_3) = 6 \tag{3.64}$$
$$S_0^1(\omega_4) = 5, \quad S_1^1(\omega_4) = 4, \quad S_2^1(\omega_4) = 3. \tag{3.65}$$

Then

$$X = \max(0, S_0^1 - 7, S_1^1 - 7, S_2^1 - 7), \tag{3.66}$$

is the value at time T of a so-called *look-back* option, where this value depends on the prices of the underlying asset S^1 in the past as well as at time T.

(a) Draw a tree to indicate the possible "paths" followed by the risky asset price process S^1.

(b) Find an equivalent martingale measure for the model.

(c) Find a replicating strategy for the option whose value at time T is given by X.

(d) What is the arbitrage free price for the option at time zero?

3. Consider a finite market model with $T = 2$, $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5\}$, and $P(\{\omega_i\}) > 0$ for $i = 1, \ldots, 5$. Suppose there are two assets, a riskless asset with price process $S^0 = \{S_t^0, t = 0, 1, 2\}$ where $S_t^0 = (1 + r)^t$ for $t = 0, 1, 2$, and some $r \geq 0$, and a risky asset with price process $S^1 = \{S_t^1, t = 0, 1, 2\}$ where

$$S_0^1(\omega_1) = 5, \quad S_1^1(\omega_1) = 8, \quad S_2^1(\omega_1) = 9 \tag{3.67}$$
$$S_0^1(\omega_2) = 5, \quad S_1^1(\omega_2) = 8, \quad S_2^1(\omega_2) = 7 \tag{3.68}$$
$$S_0^1(\omega_3) = 5, \quad S_1^1(\omega_3) = 8, \quad S_2^1(\omega_3) = 6 \tag{3.69}$$
$$S_0^1(\omega_4) = 5, \quad S_1^1(\omega_4) = 4, \quad S_2^1(\omega_4) = 6 \tag{3.70}$$
$$S_0^1(\omega_5) = 5, \quad S_1^1(\omega_5) = 4, \quad S_2^1(\omega_5) = 3. \tag{3.71}$$

Let $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_1 = \sigma\{S_0^1, S_1^1\}$ and $\mathcal{F}_2 = \sigma\{S_0^1, S_1^1, S_2^1\}$.

(a) Draw a tree to indicate the possible "paths" followed by the risky asset price process S^1.

(b) Suppose $r = 0.1$. Is there an equivalent martingale measure for this model? If there is one, is it unique? What are the answers to the last two questions if $r = 1$?