Chapter 2

Binomial Model

In this chapter we consider a simple discrete financial market model called the binomial or Cox-Ross-Rubinstein (CRR) [1] model. We derive the unique arbitrage free price for any European contingent claim based on this model. (A European contingent claim is a contingent claim that can only be exercised at the terminal time.) Existence of a no arbitrage price depends on the existence of a so-called risk neutral probability and uniqueness depends on there being a replicating strategy for the contingent claim.

2.1 Binomial or CRR Model

The CRR model is a simple discrete time model for a financial market. There are finitely many trading times $t = 0, 1, \ldots, T$ ($T < \infty$), and two assets, a risky security called a stock, and a riskless security called a bond.

The *bond* is assumed to yield a constant rate of return $r \geq 0$ over each time period $(t-1, t]$ and so assuming the bond is valued at \$1 at time zero, the value of the bond at time t is given by

$$B_t = (1 + r)^t, \quad t = 0, 1, \ldots, T. \tag{2.1}$$

The *stock* price process is modeled as an exponential random walk such that S_0 is a strictly positive constant and

$$S_t = S_{t-1}\xi_t, \quad t = 1, 2, \ldots, T, \tag{2.2}$$

where $\{\xi_t, t = 1, 2, \ldots, T\}$ is a sequence of independent and identically distributed random variables with

$$P(\xi_t = u) = p = 1 - P(\xi_t = d), \tag{2.3}$$

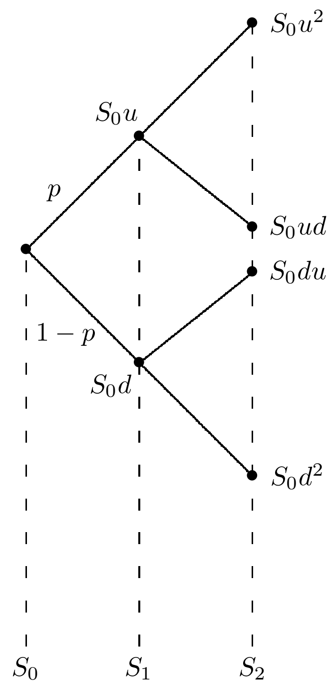

Figure 2.1: Binary tree for $T = 2$

where $p \in (0, 1)$ and $0 < d < 1 + r < u$. The latter are assumed to avoid arbitrage opportunities in the primary market model and to ensure stock prices are strictly positive. Note that

$$S_t = S_0 \prod_{i=1}^{t} \xi_i, \quad t = 0, 1, \dots, T, \tag{2.4}$$

and one may represent the possible paths that S_t follows using a binary tree (see Figure 2.1). Note that there are only three distinct values for S_2, i.e., the two middle dots have the same value for S_2. The points have been drawn as two distinct points to emphasize the fact that they may be reached by different paths, that is, through different values for the sequence S_0, S_1, S_2.

For concreteness, and without loss of generality, we assume that the probability space $(\Omega, \mathcal{F}, P)$ on which our random variables are defined is such that Ω is the finite set of 2^T possible outcomes for the values of the stock price $(T+1)$-tuple, $(S_0, S_1, S_2, \dots, S_T)$; $\mathcal{F}$ is the σ-algebra consisting of all possible subsets of Ω, and P is the probability measure on $(\Omega, \mathcal{F})$ associated with the binomial probability p. Then, for example,

$$P((S_0, S_1, \dots, S_T) = (S_0, S_0u, S_0u^2, \dots, S_0u^T)) = p^T \tag{2.5}$$

To describe the information available to the investor at time t, we introduce the σ-algebra generated by the stock prices up to and including time t, i.e, let

$$\mathcal{F}_t = \sigma\{S_0, S_1, \ldots, S_t\}, \quad t = 0, 1, \ldots, T. \tag{2.6}$$

In particular, with our concrete probability space, $\mathcal{F}_T = \mathcal{F}$.

A *trading strategy* (in the primary market) is a collection of pairs of random variables

$$H = \{(\alpha_t, \beta_t) : t = 1, 2, \ldots, T\} \tag{2.7}$$

where the random variable α_t represents the number of shares of stock to be held over the time interval $(t-1, t]$ and the random variable β_t represents the number of bonds to be held over the time interval $(t-1, t]$. We think of trading occuring at time $t-1$ to determine the portfolio holdings (α_t, β_t) until the next trading time t. To avoid strategies that anticipate the future, it is assumed that α_t, β_t are $\mathcal{F}_{t-1}$-measurable random variables for $t = 1, 2, \ldots, T$. Thus, the holdings in stock and bond over the time period $(t-1, t]$ can only depend on the stock prices observed up to and including time $t-1$. We will restrict attention here to *self-financing* trading strategies, namely, those trading strategies H such that

$$\alpha_t S_t + \beta_t B_t = \alpha_{t+1} S_t + \beta_{t+1} B_t, \quad t = 1, 2, \ldots, T-1, \tag{2.8}$$

and the investor's initial wealth is equal to

$$W_0 = \alpha_1 S_0 + \beta_1 B_0. \tag{2.9}$$

We will simply refer to these as trading strategies, rather than using the longer term self-financing trading strategies. We say that a trading strategy H represents a portfolio whose *value* at time t is given by $V_t(H)$, where

$$V_0(H) = \alpha_1 S_0 + \beta_1 B_0, \tag{2.10}$$
$$V_t(H) = \alpha_t S_t + \beta_t B_t, \quad t = 1, 2, \ldots, T. \tag{2.11}$$

An *arbitrage opportunity* (in the primary market) is a trading strategy H such that $V_0(H) = 0$, $V_T(H) \geq 0$ and $E[V_T(H)] > 0$. Note that, in the presence of the preceding conditions, the last condition is equivalent to $P(V_T(H) > 0) > 0$.

A *European contingent claim* is represented by a $\mathcal{F}_T$-measurable random variable X. The value of this contingent claim at the exercise time T is X. For example, a European call option with strike price K and expiration date T is represented by $X = (S_T - K)^+ \equiv \max\{0, S_T - K\}$. Similarly, a European put option with the same strike price and expiration date is represented by $X = (K - S_T)^+$.

A *replicating (or hedging) strategy* for a European contingent claim X is a trading strategy H such that $V_T(H) = X$. If there exists such a replicating strategy, the contingent claim is said to be *attainable* (or redundant).

2.2 Single Period Case

We first examine the single period case where $T = 1$.

We first show that there is a replicating strategy for any European contingent claim X. Given X, we seek a trading strategy $H = (\alpha_1, \beta_1)$, where α_1 and β_1 are constants, such that

$$V_1(H) \equiv \alpha_1 S_1 + \beta_1 B_1 = X. \tag{2.12}$$

Now S_1 has two possible values, $S_0 u, S_0 d$, and X is a measurable function of S_1, since it is $\mathcal{F}_1 = \sigma\{S_0, S_1\}$-measurable. Let X^u denote the value of X when $S_1 = S_0 u$ and X^d denote the value of X when $S_1 = S_0 d$. Then considering these two possible outcomes, (2.12) yields two equations for the two deterministic unknowns α_1, β_1:

$$\alpha_1 S_0 u + \beta_1(1+r) = X^u \tag{2.13}$$
$$\alpha_1 S_0 d + \beta_1(1+r) = X^d. \tag{2.14}$$

Solving for α_1, β_1 yields

$$\alpha_1 = \frac{X^u - X^d}{(u-d)S_0} \text{ ``}=\text{''} \frac{\delta X}{\delta S}, \tag{2.15}$$
$$\beta_1 = \frac{1}{1+r}\left(\frac{uX^d - dX^u}{u-d}\right). \tag{2.16}$$

The initial wealth needed to finance this strategy (sometimes called the manufacturing cost of the contingent claim) is

$$\begin{aligned} V_0 &= \alpha_1 S_0 + \beta_1 B_0 && (2.17)\\ &= \frac{1}{(1+r)(u-d)}\big((1+r-d)X^u + (u-(1+r))X^d\big) && (2.18)\\ &= \frac{1}{1+r}\big(p^* X^u + (1-p^*)X^d\big) && (2.19)\\ &= E^{p^*}[X^*], && (2.20) \end{aligned}$$

where $p^* = \frac{1+r-d}{u-d}$, $E^{p^*}[\,\cdot\,]$ denotes expectation with $p = p^*$, and $X^* = X/(1+r)$. Note that $E^{p^*}[S_1] = (1+r)S_0$ and so the discounted stock price process

$$\left\{S_0, \frac{1}{1+r}S_1\right\}$$

is a martingale under $p = p^*$ (relative to the filtration $\{\mathcal{F}_t\}$). Thus, under $p = p^*$, the average rate of return of the risky asset is the same as that of the riskless asset. For this reason p^* is called the *risk neutral probability.* It is important to realize that computing expectations under $p = p^*$ is a mathematical device. We are not assuming that the stock price actually moves according to this probability. That is, p^* may be unrelated to the subjective probability p that we associate with the binomial model for movements in the stock price.

For the next theorem, we need the notion of an arbitrage opportunity in the market consisting of the stock, bond and contingent claim. For this, we suppose that the price of the contingent claim at time zero is C_0. A *trading strategy* in stock, bond and the contingent claim is a triple $J = (\alpha_1, \beta_1, \gamma_1)$ of $\mathcal{F}_0$-measurable random variables (these will actually be constants), where α_1 represents the number of shares of stock held over $(0, 1]$, β_1 represents the number of bonds held over $(0, 1]$ and γ_1 represents the number of units of the contingent claim held over $(0, 1]$. The initial value of the portfolio associated with J is $V_0(J) = \alpha_1 S_0 + \beta_1 B_0 + \gamma_1 C_0$. The value of this portfolio at time one is $V_1(J) = \alpha_1 S_1 + \beta_1 B_1 + \gamma_1 X$. An *arbitrage opportunity* in the stock, bond, contingent claim market is a trading strategy $J = (\alpha_1, \beta_1, \gamma_1)$ such that $V_0(J) = 0$, $V_1(J) \geq 0$ and $E[V_1(J)] > 0$.

Theorem 2.2.1 $V_0 = E^{p^*}[X^*]$ *is the arbitrage free price for the European contingent claim* X.

PROOF. Let $H^* = (\alpha_1^*, \beta_1^*)$ denote the replicating strategy (in stock and bond) for the contingent claim X.

First we show that if the initial price C_0 of the contingent claim is anything other than V_0, then there is an arbitrage opportunity in the stock, bond, contingent claim market. Suppose $C_0 > V_0$. Then an investor starting with zero initial wealth could sell one option ($\gamma_1 = -1$), invest V_0 in the replicating strategy $H^* = (\alpha_1^*, \beta_1^*)$ and invest the remainder, $C_0 - V_0$, in bond. Thus, his trading strategy in stock, bond and contingent claim is $(\alpha_1^*, \beta_1^* + C_0 - V_0, -1)$. This has an initial value of zero and its value at time one is

$$\alpha_1^* S_1 + \beta_1^* B_1 + (C_0 - V_0) B_1 - X. \tag{2.21}$$

But the strategy (α_1^*, β_1^*) was chosen so that

$$\alpha_1^* S_1 + \beta_1^* B_1 = X, \tag{2.22}$$

and so it follows that the value at time one of the stock-bond-contingent claim portfolio is

$$(C_0 - V_0) B_1 > 0. \tag{2.23}$$

Thus, this represents an arbitrage opportunity. Similarly, if $C_0 < V_0$, then the investor can use the strategy $(-\alpha_1^*, -\beta_1^* + V_0 - C_0, 1)$ to create an arbitrage opportunity.

Now we show that if $C_0 = V_0$, then there is no arbitrage opportunity in the stock, bond, contingent claim market. Suppose that $J = (\alpha_1, \beta_1, \gamma_1)$ is a trading strategy in stock, bond and the contingent claim, with an initial value $V_0(J) =$

$\alpha_1 S_0 + \beta_1 + \gamma_1 C_0$ of zero and non-negative value $V_1(J)$ at time one. The value of the portfolio at time one is

$$V_1(J) = \alpha_1 S_1 + \beta_1 B_1 + \gamma_1 X \tag{2.24}$$

and so

$$\begin{aligned} E^{p^*}[V_1(J)] &= \alpha_1 E^{p^*}[S_1] + \beta_1(1+r) + \gamma_1 E^{p^*}[X] \\ &= \alpha_1(1+r)S_0 + \beta_1(1+r) + \gamma_1(1+r)C_0 \\ &= (1+r)V_0(J) \\ &= 0, \end{aligned}$$

where we have used the martingale property of the discounted stock price process under p^* and the definition of $C_0 = V_0 = E^{p^*}[X]/(1+r)$. Now, since the probability measure P^* on $\mathcal{F}_T$ associated with using $p^* \in (0,1)$ in place of p gives positive probability to both of the possible values of the non-negative random variable $V_1(J)$, it follows that $P^*(V_1(J) = 0) = 1$ and since P^* is equivalent to P on $\mathcal{F}_T$, $P(V_1(J) = 0) = 1$. Thus, there cannot be an arbitrage opportunity. □

2.3 Multi-Period Case

We now consider the general binomial model where T is any fixed positive integer. In this section p^* has the same value as in the single period case, namely, $p^* = \frac{1+r-d}{u-d}$.

We first show that there is a replicating strategy for a European contingent claim X. For this, given X, we seek a (self-financing) trading strategy $H = \{(\alpha_t, \beta_t), t = 1, \ldots, T\}$ such that

$$V_T(H) \equiv \alpha_T S_T + \beta_T B_T = X. \tag{2.25}$$

This is developed by working backwards through the binary tree.

Let $V_T = X$. Since X is an $\mathcal{F}_T$-measurable random variable, $V_T = f(S_0, S_1, \ldots, S_T)$ for some measurable function $f : \mathbb{R}^{T+1} \to \mathbb{R}$. Firstly, suppose we condition on knowing $S_0, S_1, \ldots, S_{T-1}$. Then the cost and associated trading strategy for manufacturing the contingent claim over the time period $(T-1, T]$ can be computed in a very similar manner to that for the single period model. Given $S_0, S_1, \ldots, S_{T-1}$, there are two possible values for V_T at time T, depending on whether $S_T = S_{T-1}u$ or $S_T = S_{T-1}d$. Denote these two values by V_T^u and V_T^d. In fact, $V_T^u = f(S_0, S_1, \ldots, S_{T-1}, S_{T-1}u)$ and $V_T^d = f(S_0, S_1, \ldots, S_{T-1}, S_{T-1}d)$. Note that these are $\mathcal{F}_{T-1}$ measurable random variables (the notation hides this fact, but has the advantage that it makes the formulas for the replicating strategy appear simpler). If the contingent claim is a European call option with strike price K and expiration date T, then $X = (S_T - K)^+$ and $V_T^u = (S_{T-1}u - K)^+$, $V_T^d = (S_{T-1}d - K)^+$.

Now, for any European contingent claim X, by similar analysis to that for the single period case, to ensure that $V_T(H) = X$, we obtain the following allocations for the time period $(T-1, T]$:

$$\alpha_T = \frac{V_T^u - V_T^d}{(u-d)S_{T-1}} \tag{2.26}$$

$$\beta_T = \frac{1}{(1+r)^T}\left(\frac{uV_T^d - dV_T^u}{u-d}\right) \tag{2.27}$$

and the capital required at time $T-1$ to finance these allocations in a self-financing manner is

$$V_{T-1} = \frac{1}{1+r}\left(p^* V_T^u + (1-p^*)V_T^d\right) \tag{2.28}$$

$$= \frac{1}{1+r}E^{p^*}[V_T \,|\, \mathcal{F}_{T-1}], \tag{2.29}$$

where $p^* = \frac{1+r-d}{u-d}$ and $E^{p*}[\cdot \,|\, \mathcal{F}_{T-1}]$ denotes the conditional expectation, given $\mathcal{F}_{T-1}$, when the subjective probability p is replaced by p^*.

We can find a (self-financing) trading strategy $H = \{(\alpha_t, \beta_t), t = 1, 2, \ldots, T\}$ with associated value process $\{V_t(H), t = 0, 1, \ldots, T\}$ by proceeding inductively back through the binary tree as follows: assuming values $V_{t+1}, \ldots, V_T = X$ associated with (self-financing) allocations $\{(\alpha_s, \beta_s), s = t+2, \ldots, T\}$ have been determined, given $S_0, S_1, \ldots, S_t$, the holdings α_{t+1} and β_{t+1} for the time period $(t, t+1]$ are chosen so that the value associated with these holdings at time $t+1$ is the same as that of the random variable V_{t+1}, i.e., letting V_{t+1}^u and V_{t+1}^d denote the two possible values of V_{t+1} given $S_0, S_1, \ldots, S_t$, define

$$\alpha_{t+1} = \frac{V_{t+1}^u - V_{t+1}^d}{(u-d)S_t}, \tag{2.30}$$

$$\beta_{t+1} = \frac{1}{(1+r)^{t+1}}\left(\frac{uV_{t+1}^d - dV_{t+1}^u}{u-d}\right). \tag{2.31}$$

One can readily check that the capital needed at time t to finance these holdings in a self-financing manner is

$$\begin{aligned}
V_t &= \frac{1}{1+r}E^{p^*}[V_{t+1} \,|\, \mathcal{F}_t] \\
&= \frac{1}{(1+r)^2}E^{p^*}\left[E^{p^*}[V_{t+2} \,|\, \mathcal{F}_{t+1}] \,|\, \mathcal{F}_t\right] \\
&= \frac{1}{(1+r)^2}E^{p^*}[V_{t+2} \,|\, \mathcal{F}_t] \\
&\vdots \\
&= \frac{1}{(1+r)^{T-t}}E^{p^*}[X \,|\, \mathcal{F}_t].
\end{aligned}$$

In particular,

$$V_0 = \frac{1}{(1+r)^T} E^{p^*}[X]. \tag{2.32}$$

Before giving the initial price of the contingent claim X, we note the following. For this, let $S_t^* = S_t/B_t = S_t/(1+r)^t$, $t = 0, 1, \dots, T$. The process $S^* = \{S_t^*, t = 0, 1, \dots, T\}$ is called the *discounted stock price process.*

Lemma 2.3.1 *Under $p = p^*$, $\{S_t^*, t = 0, 1, \dots, T\}$ is a martingale (relative to the filtration $\{\mathcal{F}_t\}$).*

PROOF. Clearly $S_t^* \in \mathcal{F}_t$ and S_t^* has finite mean for each t. To verify the conditional expectation property, fix $t \in \{0, 1, \dots, T-1\}$. Then, using the fact that $\mathcal{F}_t$ is generated by $S_0, S_1, \dots, S_t$ and ξ_{t+1} is independent of this sigma algebra, we have

$$E^{p^*}[S_{t+1}^* \mid \mathcal{F}_t] = \frac{1}{(1+r)^{t+1}} E^{p^*}[S_t \xi_{t+1} \mid \mathcal{F}_t] \tag{2.33}$$

$$= \frac{1}{(1+r)^{t+1}} S_t E^{p^*}[\xi_{t+1}] \tag{2.34}$$

$$= \frac{S_t^*}{1+r} (p^* u + (1-p^*)d) \tag{2.35}$$

$$= S_t^*, \tag{2.36}$$

where we have used the definition of $p^* = \frac{1+r-d}{u-d}$ to obtain the last line. □

The following theorem is the multi-period analogue of the single period Theorem 2.2.1. First we specify the notion of arbitrage in stock, bond and contingent claim to be used in the multi-period context. Let C_0 be the price charged for the contingent claim at time zero. Since we are only specifying a price for the contingent claim at time zero, trading in the contingent claim will only be allowed initially, whereas changes in the stock and bond holdings can occur at each of the times $t = 0, 1, \dots, T-1$. A *trading strategy* in stock, bond and the contingent claim, is a collection $J = \{(\alpha_t, \beta_t), t = 1, 2, \dots, T; \gamma_1\}$ where for $t = 1, 2, \dots, T$, α_t, β_t are $\mathcal{F}_{t-1}$-measurable random variables representing the holdings in stock and bond, respectively, to be held over the time interval $(t-1, t]$, and γ_1 is a $\mathcal{F}_0$-measurable random variable (actually a constant) representing the number of units of the contingent claim to be held over the time interval $(0, T]$. The trading strategy must be self-financing, i.e., its initial value is

$$V_0(J) = \alpha_1 S_0 + \beta_1 B_0 + \gamma_1 C_0, \tag{2.37}$$

and at each time $t = 1, \dots, T-1$,

$$\alpha_t S_t + \beta_t B_t = \alpha_{t+1} S_t + \beta_{t+1} B_t. \tag{2.38}$$

The last equation does not involve the contingent claim since this is not traded after time zero. The value of the portfolio at time T is

$$V_T(J) = \alpha_T S_T + \beta_T B_T + \gamma_1 X. \tag{2.39}$$

An *arbitrage opportunity* in the stock, bond and contingent claim market is a trading strategy J such that $V_0(J) = 0$, $V_T(J) \geq 0$ and $E[V_T(J)] > 0$.

Theorem 2.3.2 *Let $X^* = X/(1+r)^T$. Then $V_0 = E^{p^*}[X^*]$ is the arbitrage free initial price for the European contingent claim X.*

PROOF. The proof is very similar to that of Theorem 2.2.1. Let $H^* = \{(\alpha_t^*, \beta_t^*), t = 1, 2, \ldots, T\}$ denote the replicating strategy (in stock and bond) for the contingent claim X.

We use the existence of the replicating strategy to show that if $C_0 \neq V_0$, then there is an arbitrage opportunity in the stock, bond, contingent claim market. Suppose $C_0 > V_0$. Then an investor could sell one contingent claim initially, use V_0 of the proceeds to invest in the stock-bond replicating strategy H^*, and buy $C_0 - V_0$ additional bonds at time zero and hold them over the entire period $(0, T]$. Thus, the trading strategy is $J = \{(\alpha_t^*, \beta_t^* + C_0 - V_0), t = 1, 2, \ldots, T; \gamma_1 = -1\}$. This has initial value $V_0(J) = \alpha_1^* S_0 + \beta_1^* + C_0 - V_0 - C_0 = 0$, since $B_0 = 1$ and $V_0 = \alpha_1^* S_0 + \beta_1^*$. The value of this portfolio at time T is

$$\begin{aligned} V_T(J) &= \alpha_T^* S_T + \beta_T^* B_T + (C_0 - V_0) B_T - X & (2.40)\\ &= X + (C_0 - V_0) B_T - X = (C_0 - V_0) B_T > 0, & (2.41) \end{aligned}$$

where we have used the fact that $\{(\alpha_t^*, \beta_t^*), t = 1, 2, \ldots, T\}$ is a replicating strategy for the contingent claim X and so has value X at time T. Thus, J is an arbitrage opportunity. Similarly, if $C_0 < V_0$, the strategy $-J$ is an arbitrage opportunity.

Now suppose $C_0 = V_0$. We show that there is no arbitrage opportunity in stock, bond and contingent claim trading. Let $J = \{(\alpha_t, \beta_t), t = 1, 2, \ldots, T; \gamma_1\}$ be a trading strategy in stock, bond and contingent claim with an initial value of zero and final value $V_T(J)$ that is a non-negative random variable. Thus,

$$\begin{aligned} 0 &= V_0(J) = \alpha_1 S_0 + \beta_1 B_0 + \gamma_1 C_0, & (2.42)\\ 0 &\leq V_T(J) = \alpha_T S_T + \beta_T B_T + \gamma_1 X. & (2.43) \end{aligned}$$

Now, using the martingale property of S^* under p^*, we have

$$\begin{aligned}
&\frac{1}{(1+r)^T} E^{p^*}[V_T(J) \mid \mathcal{F}_{T-1}] \\
&= \alpha_T E^{p^*}[S_T^* \mid \mathcal{F}_{T-1}] + \beta_T + \gamma_1 E^{p^*}[X^* \mid \mathcal{F}_{T-1}] \\
&= \alpha_T S_{T-1}^* + \beta_T + \gamma_1 E^{p^*}[X^* \mid \mathcal{F}_{T-1}] \\
&= \frac{1}{(1+r)^{T-1}} (\alpha_T S_{T-1} + \beta_T B_{T-1}) + \gamma_1 E^{p^*}[X^* \mid \mathcal{F}_{T-1}] \\
&= \frac{1}{(1+r)^{T-1}} (\alpha_{T-1} S_{T-1} + \beta_{T-1} B_{T-1}) + \gamma_1 E^{p^*}[X^* \mid \mathcal{F}_{T-1}], \\
&= \alpha_{T-1} S_{T-1}^* + \beta_{T-1} + \gamma_1 E^{p^*}[X^* \mid \mathcal{F}_{T-1}],
\end{aligned}$$

where we used the fact that J is self-financing to obtain the second last equality. Taking conditional expectations with respect to $\mathcal{F}_{T-2}$ in the above and performing similar manipulations to those just executed using the martingale property of S^* and the self-financing property of J, we obtain

$$\begin{aligned}
&\frac{1}{(1+r)^T} E^{p^*}[V_T(J) \mid \mathcal{F}_{T-2}] \\
&= \alpha_{T-1} S_{T-2}^* + \beta_{T-1} + \gamma_1 E^{p^*}[X^* \mid \mathcal{F}_{T-2}] \\
&= \alpha_{T-2} S_{T-2}^* + \beta_{T-2} + \gamma_1 E^{p^*}[X^* \mid \mathcal{F}_{T-2}].
\end{aligned}$$

Applying this same procedure iteratively we finally obtain

$$\begin{aligned}
\frac{1}{(1+r)^T} E^{p^*}[V_T(J)] &= \frac{1}{(1+r)^T} E^{p^*}[V_T(J) \mid \mathcal{F}_0] \\
&= \alpha_1 S_0^* + \beta_1 + \gamma_1 E^{p^*}[X^*].
\end{aligned}$$

However, since $S_0^* = S_0$, $B_0 = 1$, and $C_0 = V_0 = E^{p^*}[X^*]$, the last line is equal to $V_0(J) = 0$. Since it was assumed that $V_T(J) \geq 0$ and the probability measure associated with p^* gives positive probability to all possible values of $V_T(J)$, it follows that $P(V_T(J) = 0) = 1$ and hence $E[V_T(J)] = 0$. Thus, there cannot be any arbitrage opportunity when $C_0 = V_0$. □

2.4 Exercises

1. Consider a single period CRR model with $S_0 = \$50$, $S_1 = \$100$ or $\$25$, $r = 0.25$.

(a) Find the arbitrage free price of a European call option for one share of stock where the strike price is $K = \$50$ and the exercise time $T = 1$.

(b) Find a hedging strategy that replicates the value of the option described in (a).

(c) Suppose the option in (a) is initially priced at \$1 above the arbitrage free price. Describe a strategy (for trading in stock, bond and the option) that is an arbitrage.

(d) What is the arbitrage free price for a put option with the same strike price and exercise time as the call option described in (a)?

2. Consider a CRR model with $T = 2$, $S_0 = \$50$, $S_1 = \$100$ or $S_1 = \$25$, and an associated European call option with strike price $K = \$40$ and exercise time $T = 2$. Assume that the risk free interest rate is $r = 0.1$.

(a) Draw the binomial tree and compute the arbitrage free price of the European call option at time zero.

(b) Determine an explicit hedging strategy for this option.

(c) (Optional) If you are ambitious, try to automate this pricing procedure in a computer program where T, S_0, u, d, K are variables.

3. (Computer lab exercise) Exercise 1 on page 76 of class handout.

Bibliography

[1] Cox, J. C., Ross, S. A., and Rubinstein, M. (1979), Option pricing: a simplified approach, *J. Finan. Econom.*, **7,** 229–263.