2.5 Pricing an American Contingent Claim

An *American contingent claim* (ACC) is represented by a (finite) sequence $Y = \{Y_t, t = 0, 1, \dots, T\}$ of real-valued random variables such that $Y_t \in \mathcal{F}_t$ for $t = 0, 1, 2, \dots, T$. The random variable Y_t, $t = 0, 1, 2, \dots, T$, is interpreted as the payoff for the claim if the owner cashes it in at time t. The time at which the owner cashes in the claim is required to be a stopping time, i.e., a random variable $\tau : \Omega \longrightarrow \{0, 1, \dots, T\}$ such that $\{\tau = t\} \in \mathcal{F}_t$, $t = 0, 1, 2, \dots, T$. For $s, t \in \{0, 1, \dots, T\}$ such that $s \leq t$, let $\mathcal{T}_{[s,t]}$ denote the set of integer-valued stopping times that take values in the interval $[s, t]$. An example of an American contingent claim is an *American call option* with strike price K which has payoff $Y_t = (S_t - K)^+$ at time t, $t = 0, 1, 2, \dots, T$. Note that if $S_t \leq K$, cashing in the contingent claim at time t is equivalent to not exercising the option at all. We have adopted this convention so that we can use one framework for treating all contingent claims, including options and contracts.

An important feature of an American contingent claim is that the buyer and the seller of such a derivative have different actions available to them — the buyer may cash in the claim at any stopping time $\tau \in \mathcal{T}_{[0,T]}$, whereas the seller seeks protection from the risk associated with all possible choices of the stopping time τ by the buyer. As with the pricing of European contingent claims, for the pricing of American contingent claims, an essential role will be played by a trading strategy that hedges the risk of the seller of an American contingent claim. However, unlike the European contingent claim setting, the seller will not always be able to exactly replicate the payoff of the American contingent claim at all times t. Instead, the seller of an American contingent claim seeks a so-called *perfect hedging* strategy which is a self-financing strategy H whose value is at least as great as the payoff of the American contingent claim at each time t.

More precisely, let $Y = \{Y_0, Y_1, \dots, Y_T\}$ be the payoff sequence for an American contingent claim. For $t = 0, 1, \dots, T$, let Z_t denote the minimum amount of wealth that the seller must have at time t in order to cover the payoff if the buyer exercises the claim at some stopping time $\tau \in \mathcal{T}_{[t,T]}$. A perfect hedging strategy for the seller is a self-financing trading strategy $H = \{(\alpha_t, \beta_t) : t = 1, 2, \dots T\}$ with value $V_t(H)$ at time $t = 0, 1, \dots, T$, such that $Z_t \leq V_t(H)$ for $t = 0, 1, 2, \dots T$. Such a H can be constructed stepwise by proceeding backwards in the binary tree. In order to see this, note that $Z_T = Y_T$. Conditioned on $\mathcal{F}_{T-1}$, let Z_T^u denote the amount that the seller must cover at time T if the value of the stock at that time is $S_{T-1}u$ and let Z_T^d denote the amount that the seller must cover at time T if the value of the stock at time T is $S_{T-1}d$. Thus, in the same manner as for the case of the European contingent claim, conditioned on $\mathcal{F}_{T-1}$, the minimum amount of wealth needed at time $T - 1$ to cover the possible payoff of the claim at time T is

$$\frac{1}{1+r} E^{p^*} \left[Z_T \mid \mathcal{F}_{T-1} \right],$$

and there exist $\mathcal{F}_{T-1}$-measurable allocations to stock and bond for the time interval $(T-1, T]$, which we denote by $(\tilde{\alpha}_T, \tilde{\beta}_T)$, such that

$$\tilde{\alpha}_T S_T + \tilde{\beta}_T B_T = Z_T, \tag{2.44}$$

$$\tilde{\alpha}_T S_{T-1} + \tilde{\beta}_T B_{T-1} = \frac{1}{1+r} E^{p^*}\left[Z_T \mid \mathcal{F}_{T-1}\right]. \tag{2.45}$$

Now,

$$Z_{T-1} = \max\left\{Y_{T-1}, \frac{1}{1+r} E^{p^*}\left[Z_T \mid \mathcal{F}_{T-1}\right]\right\},$$

so that it is possible to cover the payoff Y_{T-1} associated with the buyer cashing in the claim at time $T-1$ and to have sufficient wealth to produce a value at time T that is at least as large as the time T-payoff $Z_T = Y_T$. Let

$$\tilde{\delta}_T = Z_{T-1} - \frac{1}{1+r} E^{p^*}\left[Z_T \mid \mathcal{F}_{T-1}\right]. \tag{2.46}$$

Then $\tilde{\delta}_T$ is the excess wealth in Z_{T-1} over what is needed to cover the claim payoff at time T. Note that there is no excess if $Y_{T-1} < Z_{T-1}$. Proceeding backwards inductively through the tree and repeating a very similar argument at each stage to that applied for the time interval $(T-1, T]$, we see that for $t = 0, 1, \ldots, T-1$, conditioned on $\mathcal{F}_t$, the amount of wealth needed at time t to cover possible payoff of the claim in $[t, T]$ is

$$Z_t = \max\left\{Y_t, \frac{1}{1+r} E^{p^*}\left[Z_{t+1} \mid \mathcal{F}_t\right]\right\}.$$

Moreover, given $S_0, S_1, \ldots, S_t$, and letting Z^u_{t+1} and Z^d_{t+1} denote the two possible values of Z_{t+1} corresponding to whether $S_{t+1} = S_t u$ or $S_{t+1} = S_t d$, the allocations in stock and bond, $(\tilde{\alpha}_{t+1}, \tilde{\beta}_{t+1})$ over $(t, t+1]$, that have value

$$\frac{1}{1+r} E^{p^*}\left[Z_{t+1} \mid \mathcal{F}_t\right],$$

at time t and value Z_{t+1} at time $t+1$ are given by

$$\tilde{\alpha}_{t+1} = \frac{Z^u_{t+1} - Z^d_{t+1}}{(u-d)S_t} \tag{2.47}$$

$$\tilde{\beta}_{t+1} = \frac{1}{(1+r)^{t+1}}\left(\frac{uZ^d_{t+1} - dZ^u_{t+1}}{u-d}\right). \tag{2.48}$$

Let

$$\tilde{\delta}_{t+1} = Z_t - \frac{1}{1+r} E^{p^*}\left[Z_{t+1} \mid \mathcal{F}_t\right].$$

Then $\tilde{\delta}_{t+1}$ is the excess wealth in Z_t over what is needed to cover possible payoff of the claim in $[t+1, T]$. Note from the definition of Z_t that $\tilde{\delta}_{t+1} = 0$ if $Y_t < Z_t$.

Assuming that $Z_t, t = 0, 1, \ldots, T$ and $\tilde{\alpha}_t, \tilde{\beta}_t, \tilde{\delta}_t, t = 1, 2, \ldots, T$ are defined as above, for $t = 1, 2, \ldots, T$, let

$$\alpha_t^* = \tilde{\alpha}_t \tag{2.49}$$

$$\beta_t^* = \tilde{\beta}_t + \sum_{s=1}^{t} \frac{\tilde{\delta}_s}{B_s}. \tag{2.50}$$

Then $H^* = (\alpha_t^*, \beta_t^*), t = 1, \ldots, T\}$ is a self-financing trading strategy in stock and bond with an initial value of

$$V_0(H^*) = \alpha_1^* S_0 + \beta_1^* B_0 = Z_0$$

and a value at time $t = 1, 2, \ldots, T$ equal to

$$V_t(H^*) = \tilde{\alpha}_t S_t + \tilde{\beta}_t B_t + \sum_{s=1}^{t} \frac{\tilde{\delta}_s}{B_s} B_t.$$

Note that by construction, for $t = 1, \ldots, T$,

$$\tilde{\alpha}_t S_t + \tilde{\beta}_t B_t = Z_t$$

and so

$$V_t(H^*) = Z_t + \sum_{s=1}^{t} \frac{\tilde{\delta}_s}{B_s} B_t \geq Z_t.$$

Let

$$\tau^* = \min\{v \geq 0 : Z_v = Y_v\}. \tag{2.51}$$

Note that $\tilde{\delta}_s = 0$ for $1 \leq s \leq \tau^*$ and so

$$V_t(H^*) = Z_t \quad \text{for } 0 \leq t \leq \tau^*.$$

For $t = 0, 1, \ldots, T$, define the discounted random variables

$$Y_t^* = (1+r)^{-t} Y_t, \qquad Z_t^* = (1+r)^{-t} Z_t. \tag{2.52}$$

Then for $t = 0, 1, \ldots, T-1$,

$$Z_t^* = \max\left\{Y_t^*, E^{p^*}\left[Z_{t+1}^* \mid \mathcal{F}_t\right]\right\}, \tag{2.53}$$

and $Z_T^* = Y_T^*$.

Remark. As given by the formulas above, $\{Z_t^*, t = 0, 1, \ldots, T\}$ is called the *Snell envelope* of $\{Y_t^*, t = 0, 1, \ldots, T\}$.

Theorem 2.5.1

(i) $\{Z_t^*, \mathcal{F}_t, t = 0, 1, \ldots, T\}$ *is the smallest supermartingale (under* p^**) such that* $Z_t^* \geq Y_t^*$ *for* $t = 0, 1, \ldots, T$,

(ii) $Z_t^* = \max_{\tau \in \mathcal{T}_{[t,T]}} E^{p^*}[Y_\tau^* \mid \mathcal{F}_t]$, $t = 0, 1, \ldots, T$,

(iii) $\tau^*(t) \equiv \min\{v \geq t : Z_v^* = Y_v^*\}$ *achieves the maximum in the right side of the equality in (ii) for* $t = 0, 1, \ldots, T$.

Proof. Throughout this proof, all expectations and conditional expectations are to be computed under p^*. For (i), the supermartingale property and inequality are immediate consequences of the definition of Z^*. To show that Z^* is the smallest supermartingale satisfying the inequality, suppose that $W = \{W_t, t = 0, 1, \ldots, T\}$ is another supermartingale such that $W_t \geq Y_t^*$, $t = 0, 1, \ldots, T$. Then $W_T \geq Y_T^* = Z_T^*$. For a proof by backwards induction, suppose that for some $t \in \{0, 1, \ldots, T-1\}$, $W_{t+1} \geq Z_{t+1}^*$. Then by the supermartingale property of W,

$$W_t \geq E^{p^*}[W_{t+1} \mid \mathcal{F}_t] \geq E^{p^*}\left[Z_{t+1}^* \mid \mathcal{F}_t\right].$$

Since, by assumption $W_t \geq Y_t^*$, W_t is greater than or equal to the maximum of Y_t^* and $E^{p^*}[Z_{t+1}^* \mid \mathcal{F}_{t+1}]$, which equals Z_t^*. This completes the induction step and so $W_t \geq Z_t^*$ for $t = T, T-1, \ldots, 1, 0$.

We prove (ii) and (iii) together, using backwards induction again. For $t = T$, both (ii) and (iii) are easy to show since $\mathcal{T}_{[T,T]} = T$ and $Z_T^* = Y_T^*$. For the induction step, assume that for some $s \in \{0, 1, \ldots, T-1\}$ both (ii) and (iii) hold for $t = s+1, s+2, \ldots, T$. By the definition of Z_s^* and (ii) for $t = s+1$, for each $\tau \in \mathcal{T}_{[s+1,T]}$ we have

$$Z_s^* \geq \max\left\{Y_s^*, E^{p^*}\left[E^{p^*}[Y_\tau^* \mid \mathcal{F}_{s+1}] \mid \mathcal{F}_s\right]\right\} = \max\left\{Y_s^*, E^{p^*}[Y_\tau^* \mid \mathcal{F}_s]\right\}. \tag{2.54}$$

For $\sigma \in \mathcal{T}_{[s,T]}$,

$$Y_\sigma^* = 1_{\{\sigma = s\}} Y_s^* + 1_{\{\sigma \geq s+1\}} Y_{\sigma \vee (s+1)}^*.$$

Since $1_{\{\sigma \geq s+1\}} \in \mathcal{F}_s$,

$$\begin{aligned} E^{p^*}[Y_\sigma^* \mid \mathcal{F}_s] &= 1_{\{\sigma = s\}} Y_s^* + 1_{\{\sigma \geq s+1\}} E^{p^*}\left[Y_{\sigma \vee (s+1)}^* \mid \mathcal{F}_s\right] \\ &\leq \max\left\{Y_s^*, E^{p^*}\left[Y_{\sigma \vee (s+1)}^* \mid \mathcal{F}_s\right]\right\}. \end{aligned} \tag{2.55}$$

Since $\sigma \vee (s+1) \in \mathcal{T}_{[s+1,T]}$, it follows from (2.54) and (2.55) that

$$E^{p^*}[Y_\sigma^* \mid \mathcal{F}_s] \leq Z_s^*,$$

which proves that

$$Z_s^* \geq \max_{\tau \in \mathcal{T}_{[s,T]}} E^{p^*}[Y_\tau^* \mid \mathcal{F}_s]. \tag{2.56}$$

The proof of (ii) and (iii) for $t = s$ will be complete once we verify that the maximum in the right member of (2.56) is achieved at $\tau = \tau^*(s)$ and the value of this maximum is Z_s^*. By the definition of Z_s^* and (iii) for $t = s+1$,

$$\begin{aligned} Z_s^* &= \max\left\{Y_s^*, E^{p^*}\left[E^{p^*}\left[Y^*_{\tau^*(s+1)} \mid \mathcal{F}_{s+1}\right] \mid \mathcal{F}_s\right]\right\} \\ &= \max\left\{Y_s^*, E^{p^*}\left[Y^*_{\tau^*(s+1)} \mid \mathcal{F}_s\right]\right\}. \end{aligned} \tag{2.57}$$

On $\{\tau^*(s) = s\}$, $Y_s^* = Z_s^*$, and on $\{\tau^*(s) \geq s+1\}$, $Y_s^* < Z_s^*$. Thus,

$$Z_s^* = 1_{\{\tau^*(s)=s\}} Y_s^* + 1_{\{\tau^*(s)\geq s+1\}} E^{p^*}\left[Y^*_{\tau^*(s+1)} \mid \mathcal{F}_s\right].$$

Moreover, on $\{\tau^*(s) \geq s+1\}$, $\tau^*(s) = \tau^*(s+1)$ and so the above yields

$$Z_s^* = E^{p^*}\left[Y^*_{\tau^*(s)} \mid \mathcal{F}_s\right],$$

which completes the proof of (ii) and (iii) for $t = s$. □

Recall the definition of τ^* from (2.51) and note that

$$\tau^* = \tau^*(0).$$

We now argue that Z_0 is the unique arbitrage free initial price for the American contingent claim. For this, we need the notion of an arbitrage in a market where the stock and bond can be traded, and the American contingent claim (ACC) can be bought or sold at time zero. For such a market, let the initial price of the ACC be a constant C_0. There are two types of arbitrage opportunities, one for a seller and another for a buyer of the ACC. The *seller* of the ACC has an *arbitrage opportunity* if there is a self-financing trading strategy H^s in stock and bond such that $V_0(H^s) = C_0$ and for all stopping times $\tau \in \mathcal{T}_{[0,T]}$,

$$V_\tau(H^s) - Y_\tau \geq 0 \qquad \text{and} \qquad E\left[V_\tau(H^s) - Y_\tau\right] > 0. \tag{2.58}$$

The *buyer* of the ACC has an *arbitrage opportunity* if there is a self-financing trading strategy H^b such that $V_0(H^b) = -C_0$ and there exists a stopping time $\tau \in \mathcal{T}_{[0,T]}$ such that

$$V_\tau(H^b) + Y_\tau \geq 0 \qquad \text{and} \qquad E\left[V_\tau(H^b) + Y_\tau\right] > 0. \tag{2.59}$$

The price C_0 is arbitrage free if there is no arbitrage opportunity for a seller or buyer of the contingent claim at this price.

To take advantage of a seller's arbitrage opportunity, an investor could sell one ACC at time zero for C_0 and invest the proceeds C_0 according to the trading strategy H^s until the claim is cashed in by the buyer at some stopping time τ. At time τ, the seller would give the buyer Y_τ to payoff the claim and could put the remainder of his wealth $V_\tau(H^s) - Y_\tau$ in bond for the time period $(\tau, T]$. This would result in a final value of $(V_\tau(H^s) - Y_\tau)(1+r)^{T-\tau}$ which is non-negative and is strictly positive with positive probability.

To take advantage of a buyer's arbitrage opportunity, an investor could buy one ACC at time zero for C_0, and invest $-C_0$ according to H^b until the time τ when the buyer cashes in the claim. The buyer would then have $V_\tau(H^b) + Y_\tau$ at time τ and could put this in bond for the time period $(\tau, T]$ so that the buyer's final wealth would be $(V_\tau(H^b) + Y_\tau)(1+r)^{T-\tau}$ which is non-negative and is strictly positive with positive probability.

The following lemma will used in showing that an initial price of Z_0 for the ACC is arbitrage free.

Lemma 2.5.2 *Let H be a self-financing trading strategy in stock and bond with value process $\{V_t(H), t = 0, 1, \ldots, T\}$ and discounted value process $\{V_t^*(H) = V_t(H)/(1+r)^t, t = 0, 1, \ldots, T\}$. Then $\{V_t^*(H), \mathcal{F}_t, t = 0, 1, \ldots, T\}$ is a martingale under p^*. In particular, for any $\tau \in \mathcal{T}_{[0,T]}$,*

$$E^{p^*}[V_\tau^*(H)] = V_0^*(H). \tag{2.60}$$

Proof. Let $H = \{(\alpha_t, \beta_t), t = 0, 1, \ldots, T\}$. Note that $V_t^*(H) = \alpha_t S_t^* + \beta_t$ and $\alpha_t, \beta_t \in \mathcal{F}_{t-1}$ for $t = 1, \ldots, T$. Also recall that $\{S_t^*, \mathcal{F}_t, t = 0, 1, \ldots, T\}$ is a martingale under p^*. Therefore, for $t = 1, \ldots, T$,

$$E^{p^*}[V_t^*(H) \mid \mathcal{F}_{t-1}] = \alpha_t E^{p^*}[S_t^* \mid \mathcal{F}_{t-1}] + \beta_t \tag{2.61}$$

$$= \alpha_t S_{t-1}^* + \beta_t. \tag{2.62}$$

By factoring out $1/(1+r)^{t-1}$ and using the self-financing property of H, it follows that for $t = 1, \ldots, T$

$$E^{p^*}[V_t^*(H) \mid \mathcal{F}_{t-1}] = \frac{1}{(1+r)^{t-1}}(\alpha_t S_{t-1} + \beta_t B_{t-1}) \tag{2.63}$$

$$= \frac{1}{(1+r)^{t-1}}(\alpha_{t-1} S_{t-1} + \beta_{t-1} B_{t-1}) \tag{2.64}$$

$$= V_{t-1}^*(H). \tag{2.65}$$

Thus, $\{V_t^*(H), \mathcal{F}_t, t = 0, 1, \ldots, T\}$ is a martingale under p^*. Equation (2.60) follows from Doob's stopping theorem since τ is a bounded stopping time. □

Theorem 2.5.3 *The unique arbitrage free price at time zero for the American contingent claim is Z_0.*

Proof. We first show that the arbitrage free price cannot be anything other than Z_0, i.e., we establish uniqueness of an arbitrage free initial price.

Suppose that $C_0 > Z_0$. Then there is an arbitrage opportunity for the seller of the American contingent claim. To see this, let H^s denote the self-financing trading strategy in stock and bond corresponding to investing Z_0 according to the perfect hedging strategy H^* and $C_0 - Z_0$ in bonds for all time. Note that the initial value $V_0(H^s) = C_0$ and the value $V_t(H^*)$ of H^* at time t is at least Z_t for $t = 0, 1, \ldots, T$. Then for any stopping time $\tau \in \mathcal{T}_{[0,T]}$,

$$\begin{aligned} V_\tau(H^s) - Y_\tau &= V_\tau(H^*) + (C_0 - Z_0)B_\tau - Y_\tau && (2.66)\\ &\geq Z_\tau + (C_0 - Z_0)B_\tau - Y_\tau && (2.67)\\ &\geq (C_0 - Z_0)B_\tau, && (2.68) \end{aligned}$$

where $(C_0 - Z_0)B_\tau > 0$. Thus, there is an arbitrage opportunity for a seller of the American contingent claim.

On the other hand, suppose that $C_0 < Z_0$. Then there is an arbitrage opportunity for the buyer of the American contingent claim. To see this, let H^b denote the self-financing trading strategy corresponding to investing $-Z_0$ according to the negative $-H^*$ of the perfect hedging strategy H^* and investing $Z_0 - C_0$ in bonds for all time. Furthermore, consider the stopping time τ^* (viewed as the time at which the ACC should be cashed in). Note that $V_t(-H^*) = -V_t(H^*)$ for $t = 0, 1, \ldots, T$, and $V_0(H^b) = -C_0$. Then, using the fact that $V_{\tau^*}(H^*) = Z_{\tau^*} = Y_{\tau^*}$, we have

$$V_{\tau^*}(H^b) + Y_{\tau^*} = -V_{\tau^*}(H^*) + (Z_0 - C_0)B_{\tau^*} + Y_{\tau^*} = (Z_0 - C_0)B_{\tau^*}.$$

Since $(Z_0 - C_0)B_{\tau^*} > 0$, there is an arbitrage opportunity for a buyer of the American contingent claim.

Finally, suppose that $C_0 = Z_0$. We need to show that $C_0 = Z_0$ is arbitrage free, i.e., that there exists an arbitrage free initial price. We begin by showing that there is no arbitrage opportunity for a seller of an American contingent claim with $C_0 = Z_0$. For a proof by contradiction, suppose that there exists a self-financing trading strategy H^s such that $V_0(H^s) = Z_0$ and for each $\tau \in \mathcal{T}_{[0,T]}$, (2.58) holds. Note that $\tau^* \in \mathcal{T}_{[0,T]}$. Then it follows from (2.58) with $\tau = \tau^*$ and the equivalence of the probability measures under p and p^*, that

$$E^{p^*}\left[V^*_{\tau^*}(H^s) - Y^*_{\tau^*}\right] > 0. \qquad (2.69)$$

On the other hand, by (2.60), $E^{p^*}[V^*_{\tau^*}(H^s)] = V^*_0(H^s) = Z^*_0$, and by (ii) and (iii) in Theorem 2.5.1, $E^{p^*}[Y^*_{\tau^*}] = Z^*_0$. Combining these two properties, we obtain $E^{p^*}[V^*_{\tau^*}(H^s) - Y^*_{\tau^*}] = 0$, which contradicts (2.69). Therefore, no such H^s exists and consequently there is no arbitrage opportunity for a seller of the American contingent claim.

Next we show that there is no arbitrage opportunity for a buyer of the American contingent claim with $C_0 = Z_0$. For this, suppose that there exists a self-financing trading strategy H^b such that $V_0(H^b) = -Z_0$ and a stopping time $\tau \in \mathcal{T}_{[0,T]}$ such that (2.59) holds. For a contradiction, we will show that $E\left[V_\tau(H) + Y_\tau\right] \leq 0$, or equivalently that $E^{p^*}\left[V_\tau(H) + Y_\tau\right] \leq 0$, which in turn is equivalent to showing that $E^{p^*}\left[V^*_\tau(H) + Y^*_\tau\right] \leq 0$ (since we have assumed that

$V_\tau(H^b) - Y_\tau \geq 0)$. By (2.60), $E^{p^*}\left[V_\tau^*(H^b)\right] = V_0^*(H^b) = -Z_0^*$. Moreover, by (ii) in Theorem 2.5.1, $E^{p^*}\left[Y_\tau^*\right] \leq Z_0^*$. Therefore, $E^{p^*}\left[V_\tau^*(H^b) + Y_\tau^*\right] \leq 0$, and so no such pair H^b, τ exists. Consequently there is no arbitrage opportunity for a buyer of the American contingent claim. □