Math 194: Homework 4, due Tuesday February 6, 2001.

1. This problem is a continuation of problem 1 from homework 3. Suppose that the American put option is priced at \$1 less than the arbitrage free price at time zero. Explicitly describe an arbitrage opportunity for a buyer of the American put option and verify that this indeed represents an arbitrage opportunity.

2. Consider a stock price process $S = (S_0, S_1, S_2)$ defined on $\Omega = \{\omega_1, \omega_2, \ldots, \omega_6\}$ by

$$S(\omega_1) = (8, 16, 32)$$
$$S(\omega_2) = (8, 16, 28)$$
$$S(\omega_3) = (8, 16, 10)$$
$$S(\omega_4) = (8, 4, 10)$$
$$S(\omega_5) = (8, 4, 4)$$
$$S(\omega_6) = (8, 4, 2)$$

Note, this is not a binomial model. However, you can draw a tree to describe the possible paths followed by the stock price process. You should do this and then answer the following questions.

(a) For $t = 0, 1, 2$, let $\mathcal{F}_t = \sigma\{S_s : 0 \leq s \leq t\}$, the σ-algebra generated by the stock price process up to time t. The entire collection of σ-algebras $\{\mathcal{F}_t, t = 0, 1, 2\}$ is called a filtration. Each $\mathcal{F}_t$ is generated by a partition $\mathcal{P}_t$ of Ω. Determine what $\mathcal{P}_t$ is for $t = 0, 1, 2$. Write down the list of sets that constitute $\mathcal{F}_t$, for $t = 0, 1, 2$.

(b) Let P be a probability measure defined on $(\Omega, \mathcal{F}_2)$ where $P(\{\omega_1\}) = 0.5$, $P(\{\omega_2\}) = 0.2$, $P(\{\omega_3\}) = 0.1$, $P(\{\omega_4\}) = 0.1$ $P(\{\omega_5\}) = 0.05$, $P(\{\omega_6\}) = 0.05$. Let E denote expectation under P. Determine the conditional expectations:

$$E[S_2|\mathcal{F}_1],$$
$$E[S_1|\mathcal{F}_0].$$

(c) Let $\tau_1 = \min\{t \geq 0 : S_t \geq 15\} \wedge 2$. Write out what τ_1 is as a function on Ω. Prove that τ_1 is a stopping time.

(d) Let $\tau_2 = \max\{t \geq 0 : S_t > 4\}$. Write out what τ_2 is as a function on Ω. Show that τ_2 is *not* a stopping time.

3. This exercise will require you to obtain historical data on the daily closing price of America Online stock. Such data can be readily obtained from the Yahoo finance web site (a link to this is available from the Math 194 web site).

Consider a binomial model with $p = 0.5$, i.e., the stock price process for $t = 0, 1, \ldots, T$ is given by

$$S_t = S_{t-1}\xi_t, \quad t = 1, \ldots, T,$$

where S_0 is a positive constant, $\{\xi_t, t = 1, \ldots, T\}$ is a sequence of i.i.d. random variables with

$$P(\xi_t = u) = p, \quad P(\xi_t = d) = 1 - p,$$

$0 < d < 1 + r < u$, and r is the risk-free interest rate. Let

$$X_t = \frac{S_t - S_{t-1}}{S_{t-1}} = \xi_t - 1, \quad t = 1, \ldots, T.$$

Then, with $p = 0.5$, the mean and variance of X_t are given by

$$\mu \quad = \quad E^p[X_t] = pu + (1-p)d - 1 = \frac{u+d}{2} - 1, \tag{1}$$

$$Var \quad = \quad E^p[(X_t - \mu)^2] = \frac{(u-d)^2}{4}, \tag{2}$$

respectively.

Suppose that you have a sample of closing stock prices $\tilde{S}_0, \tilde{S}_1, \ldots, \tilde{S}_T$ for times $t = 0, 1, \ldots, T$. For $t = 0, 1, \ldots, T$, let

$$\tilde{X}_t = \frac{\tilde{S}_t - \tilde{S}_{t-1}}{\tilde{S}_{t-1}}.$$

Then, the usual (unbiased) statistical estimators for μ and Var are given by

$$\bar{\mu} = \frac{1}{T}\sum_{t=1}^{T} \tilde{X}_t, \qquad \overline{Var} = \frac{1}{T-1}\sum_{t=1}^{T}(\tilde{X}_t - \bar{\mu})^2.$$

(a) Compute these estimators using the daily closing stock price data for America Online (ticker symbol AOL), commencing with December 3, 2000 and ending with January 3, 2001.

You can obtain this historical data in downloadable Excel spreadsheet form from the Yahoo finance website http://finance.yahoo.com/. Once there, type in the ticker symbol AOL, select "Chart" and click on "Get Quotes", at the bottom of the chart area click on "historical quotes ", type in the range of dates for which you want data and select "Daily", then click on "Get Historical Data". The historical data for the range of dates will be displayed and can be saved in spreadsheet format as a .csv file by clicking on "Download Spreadsheet Format" at the bottom of the table. Once you have saved the data to a file, you can open that file using Excel (note that it has a .csv extension, not a .xls extension). You can then use the built in Excel functions Average and Var to compute the above estimators for a column of data – you can also use Stdev instead of Var to obtain the square root of Var. Here, the natural choice for one unit of time is one trading day (ignore non-trading days such as weekends and holidays in counting time).

Using these estimates in place of μ and Var in (1) and (2) respectively, determine estimates for u and d.

(b) Now consider the binomial model with these estimated values of u and d.

Based on this binomial model, find the arbitrage free price on January 3, 2001 for

(i) a European call option based on AOL stock with a strike price of $20 and an expiration month of January,

(ii) an American put option based on AOL stock with a strike price of $20 and an expiration month of January.

Notes:

(N1) To estimate the daily risk-free interest rate r, use the 3-month T-bill rate on January 3, 2001 of 5.68% per annum. As an approximation, assume there are $65 = 13 \times 5$ trading days in a 3-month period, set $(1+r)^{65} = 1 + \frac{0.0568}{4}$ and solve for r. (You should get $r = 0.000217$.)

(N2) Recall that all market-traded options (such as we are considering) effectively expire on the third Friday of their month of expiry (we say effectively because options cannot be traded on a Saturday). For January 2001, the third Friday was January 19, 2001. Since Martin Luther King day – January 15, 2001, was a holiday, starting with January 3, 2001 as t=0, the terminal time, January 19, 2001, will be T=11.)

(N3) To compute the arbitrage free prices, you will probably want to use an Excel spreadsheet program, such as you have been working with on previous homework assignments. You will need to draw a bigger binomial tree with more steps in it for this – in particular, you will need to start further to the right in order to have room for your tree to spread.)

(c) Comment on your findings, especially in comparison to the option prices listed in the Wall Street Journal for AOL corresponding to the close of trading on January 3, 2001 (see the handout from the first lecture): a Call option with a strike price of $20 and expiration in January was priced at $13 and the Put option with the same strike price and expiration month was priced at $0.25.