

# Math 294 Homework 4 Solutions

Professor R. J. Williams

March 2, 2018

**Problem 1** (Chapter 3, Exercise 3). Consider a finite market model with  $T = 2$ ,  $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5\}$ , and  $P(\{\omega_i\}) > 0$  for  $i = 1, \dots, 5$ . Suppose there are two assets, a riskless asset with price process  $S^0 = \{S_t^0, t = 0, 1, 2\}$  where  $S_t^0 = (1 + r)^t$  for  $t = 0, 1, 2$ , and some  $r \geq 0$ , and a risky asset with price process  $S^1 = \{S_t^1, t = 0, 1, 2\}$  where

$$\begin{aligned} S^1(\omega_1) &= (5, 8, 9), \\ S^1(\omega_2) &= (5, 8, 7), \\ S^1(\omega_3) &= (5, 4, 6), \\ S^1(\omega_4) &= (5, 4, 5), \\ S^1(\omega_5) &= (5, 4, 2). \end{aligned}$$

Let  $\mathcal{F}_0 = \{\emptyset, \Omega\}$ ,  $\mathcal{F}_1 = \sigma\{S_0^1, S_1^1\}$  and  $\mathcal{F}_2 = \sigma\{S_0^1, S_1^1, S_2^1\}$ .

- (a) Draw a tree to indicate the possible “paths” followed by the risky asset price process  $S^1$ .
- (b) Suppose  $r = 0.1$ . Is there an equivalent martingale measure for this model? If there is one, is it unique? If there is not one, demonstrate an arbitrage opportunity. What are the answers to the last three questions if  $r = 1$ ?

*Proof.* (a) See the following page.

- (b) Suppose  $P^*$  is an EMM. Then  $P^*$  satisfies  $P^*(\{\omega_i\}) > 0$  for  $i = 1, \dots, 5$  and

$$E^{P^*}[S_t^{*,1}|\mathcal{F}_{t-1}] = S_{t-1}^{*,1}, \text{ for } t = 1, 2. \quad (1)$$

Suppose  $r = 0.1$  and there exists an EMM  $P^*$ . Set  $t = 1$ . Then the left-hand-side of (1) is given by

$$\begin{aligned} E^{P^*}[S_1^{*,1}|\mathcal{F}_0] &= \frac{\$8}{1.1}P^*(\{\omega_1, \omega_2\}) + \frac{\$4}{1.1}P^*(\{\omega_3, \omega_4, \omega_5\}) \\ &= \frac{\$8}{1.1}P^*(\{\omega_1, \omega_2\}) + \frac{\$4}{1.1}(1 - P^*(\{\omega_1, \omega_2\})) \\ &= \frac{\$4}{1.1}P^*(\{\omega_1, \omega_2\}) + \frac{\$4}{1.1}. \end{aligned}$$

The right-hand-side of (1) is given by  $S_0^{*,1} = \$5$ . Setting the left- and right-hand-sides equal and solving, we get  $P^*(\{\omega_1, \omega_2\}) = 3/8$  and  $P^*(\{\omega_3, \omega_4, \omega_5\}) = 5/8$ . Now set  $t = 2$ . On  $\{\omega_1, \omega_2\}$  the left-hand-side of (1) is given by

$$\begin{aligned} E^{P^*}[S_2^{*,1}|\mathcal{F}_1] &= \frac{\frac{\$9}{1.1^2}P^*(\{\omega_1\}) + \frac{\$7}{1.1^2}P^*(\{\omega_2\})}{P^*(\{\omega_1, \omega_2\})} \\ &= \frac{\frac{\$9}{1.1^2}P^*(\{\omega_1\}) + \frac{\$7}{1.1^2}(3/8 - P^*(\{\omega_1\}))}{3/8} \\ &= \frac{8}{3} \frac{\$2}{1.1^2}P^*(\{\omega_1\}) + \frac{\$7}{1.1^2}. \end{aligned}$$

On  $\{\omega_1, \omega_2\}$ , the right-hand-side of (1) is given by  $S_1^{*,1} = \frac{\$8}{1.1}$ . Setting the left- and right-hand-sides equal and solving, we get  $P^*(\{\omega_1\}) = 27/80$  and  $P^*(\{\omega_2\}) = 3/80$ . On  $\{\omega_3, \omega_4, \omega_5\}$  the left-hand-side of (1) is given by

$$\begin{aligned} E^{P^*}[S_2^{*,1}|\mathcal{F}_1] &= \frac{\frac{\$6}{1.1^2}P^*(\{\omega_3\}) + \frac{\$5}{1.1^2}P^*(\{\omega_4\}) + \frac{\$2}{1.1^2}P^*(\{\omega_5\})}{P^*(\{\omega_3, \omega_4, \omega_5\})} \\ &= \frac{\frac{\$6}{1.1^2}(5/8 - P^*(\{\omega_4, \omega_5\})) + \frac{\$5}{1.1^2}P^*(\{\omega_4\}) + \frac{\$2}{1.1^2}P^*(\{\omega_5\})}{5/8} \\ &= \frac{\$6}{1.1^2} - \frac{8}{5} \frac{\$1}{1.1^2}P^*(\{\omega_4\}) - \frac{8}{5} \frac{\$4}{1.1^2}P^*(\{\omega_5\}). \end{aligned}$$

The right-hand-side of (1) is given by  $S_1^{*,1} = \frac{\$4}{1.1}$ . Setting the left- and right-hand-sides equal and solving, we get  $P^*(\{\omega_3\}) = -3/8 + 3P^*(\{\omega_5\})$ ,  $P^*(\{\omega_4\}) = 1 - 4P^*(\{\omega_5\})$ , and  $1/8 < P^*(\{\omega_5\}) < 1/4$ . Therefore, there exists an EMM  $P^*$  where  $P^*$  satisfies

$$\begin{aligned} P^*(\{\omega_1\}) &= 27/80, \\ P^*(\{\omega_2\}) &= 3/80, \\ P^*(\{\omega_3\}) &= 3p - 3/8, \\ P^*(\{\omega_4\}) &= 1 - 4p, \\ P^*(\{\omega_5\}) &= p, \end{aligned}$$

where  $1/8 < p < 1/4$ . Since any EMM must satisfy the above properties, this describes all possible EMMs.

Now suppose  $r = 1$  and there exists an EMM  $P^*$ . Then at  $t = 1$ , the left-hand-side of (1) is given by

$$\begin{aligned} E^{P^*}[S_1^{*,1}|\mathcal{F}_0] &= \frac{\$8}{2}P^*(\{\omega_1, \omega_2\}) + \frac{\$4}{2}P^*(\{\omega_3, \omega_4, \omega_5\}) \\ &= \frac{\$8}{2}P^*(\{\omega_1, \omega_2\}) + \frac{\$4}{2}(1 - P^*(\{\omega_3, \omega_4, \omega_5\})) \\ &= \$2P^*(\{\omega_1, \omega_2\}) + \$2 \end{aligned}$$

The right-hand-side of (1) is given by  $S_0^{*,1} = \$5$ . Solving, we get  $P^*(\{\omega_1, \omega_2\}) = 3/2 > 1$ , a contradiction. Thus, there does not exist an equivalent martingale measure. To show there exists an arbitrage opportunity, consider the trading strategy  $\phi = \{(\phi_t^0, \phi_t^1), t = 1, 2\}$ , where  $(\phi_1^0, \phi_1^1) = (5, -1)$  and  $(\phi_2^0, \phi_2^1) = (5 - \frac{1}{2}S_1^1, 0)$ . At  $t = 0$ ,

$$V_0(\phi) = \phi_1^0 S_0^0 + \phi_1^1 S_0^1 = 5(\$1) - 1(\$5) = 0.$$

At  $t = 1$ ,

$$\phi_1^0 S_1^0 + \phi_1^1 S_1^1 = 5(\$2) - S_1^1.$$

and

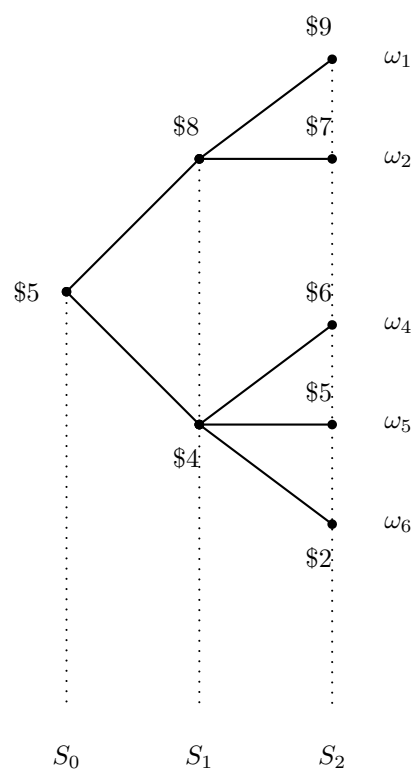
$$\phi_2^0 S_1^0 + \phi_2^1 S_1^1 = \left(5 - \frac{1}{2}S_1^1\right) \$2.$$

Therefore  $V_1(\phi) = 5(\$2) - S_1^1$  and  $\phi$  is self-financing. Note that at  $t = 1$ ,  $V_1(\phi) = \$10 - S_1^1 \geq \$10 - \$8 = \$2$ . Since  $V_1(\phi) > 0$  with probability 1, our trading strategy is to invest all our assets in the riskless asset after time one. Then at  $t = 2$

$$V_2(\phi) = \phi_2^0 S_2^0 + \phi_2^1 S_2^1 = \left(5 - \frac{1}{2}S_1^1\right) (\$4) \geq \$20 - \$2(8) = \$4.$$

Since  $V_2(\phi) > 0$  with probability 1, this is an arbitrage opportunity. □

Figure 1: Tree for Problem 1.



**Problem 2** (Chapter 3, Exercise 4). Let  $T = 1$ ,  $S_t^0 = (1+r)^t$  where  $r = 1/3$ ,  $t = 0, 1$ ,  $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ , and  $P(\{\omega_i\}) > 0$  for  $i = 1, 2, 3, 4$ . Suppose that there are two risky assets with price processes  $S^1, S^2$ :

$$\begin{aligned} S^1(\omega_1) &= (2, 4), & S^2(\omega_1) &= (5, 8), \\ S^1(\omega_2) &= (2, 4), & S^2(\omega_2) &= (5, 16/3), \\ S^1(\omega_3) &= (2, 8/3), & S^2(\omega_3) &= (5, 16/3), \\ S^1(\omega_4) &= (2, 4/3), & S^2(\omega_4) &= (5, 8). \end{aligned}$$

(a) Specify the space

$$L = \{G_T^*(\phi) : \phi \text{ is a trading strategy with } V_0(\phi) = 0\}$$

for this example.

(b) Are there any arbitrage opportunities for this example? If so, find all of them.

(c) Are there any equivalent martingale measures for this example? If so, find all of them.

*Proof.* (a) First note that the discounted stock price processes  $S^{*,1}, S^{*,2}$  are given by

$$\begin{aligned} S^{*,1}(\omega_1) &= (2, 3), & S^{*,2}(\omega_1) &= (5, 6), \\ S^{*,1}(\omega_2) &= (2, 3), & S^{*,2}(\omega_2) &= (5, 4), \\ S^{*,1}(\omega_3) &= (2, 2), & S^{*,2}(\omega_3) &= (5, 4), \\ S^{*,1}(\omega_4) &= (2, 1), & S^{*,2}(\omega_4) &= (5, 6). \end{aligned}$$

Therefore the processes  $\Delta S^{*,1}, \Delta S^{*,2}$  are given by

$$\begin{aligned} \Delta S^{*,1}(\omega_1) &= 1, & \Delta S^{*,2}(\omega_1) &= 1, \\ \Delta S^{*,1}(\omega_2) &= 1, & \Delta S^{*,2}(\omega_2) &= -1, \\ \Delta S^{*,1}(\omega_3) &= 0, & \Delta S^{*,2}(\omega_3) &= -1, \\ \Delta S^{*,1}(\omega_4) &= -1, & \Delta S^{*,2}(\omega_4) &= 1. \end{aligned}$$

Therefore, if we consider  $\Delta S^{*,1}, \Delta S^{*,2}$  as points in  $\mathbb{R}^4$  where the  $i$ th coordinate corresponds to the value for outcome  $\omega_i$ ,  $i = 1, 2, 3, 4$ . Then  $\Delta S^{*,1}$  and  $\Delta S^{*,2}$  are associated with the points  $(1, 1, 0, -1)$  and  $(1, -1, -1, 1)$  respectively. If  $\phi = (\phi^0, \phi^1, \phi^2)$  is our trading strategy, then the terminal discounted gain is given by

$$G_1^*(\phi) = \phi^1 \Delta S^{*,1} + \phi^2 \Delta S^{*,2} = (\phi^1 + \phi^2, \phi^1 - \phi^2, -\phi^2, -\phi^1 + \phi^2),$$

where the  $i$ th coordinate of  $G_1^*(\phi)$  corresponds to the terminal discounted gain for outcome  $\omega_i$ . Note that  $\phi$  is only dependent on the information at time  $t = 0$ . Therefore, if  $L = \{G_1^*(\phi) : \phi \text{ is a trading strategy such that } V_0(\phi) = 0\}$ , then  $L$  is a 2-dimensional linear subspace of  $\mathbb{R}^4$  given by

$$L = \{(\phi^1 + \phi^2, \phi^1 - \phi^2, -\phi^2, -\phi^1 + \phi^2) : \phi^1, \phi^2 \in \mathbb{R}\} = \{(x + y, x - y, -y, -x + y) : x, y \in \mathbb{R}\}.$$

Here  $\phi^1$  and  $\phi^2$  are allowed to vary over all real numbers because we can always offset the amount invested in risky assets by investing an equal and opposite amount in bonds at time zero. The discounted bonds process does not appear in our discounted gains process because the discounted bonds process is constant in time.

- (b) If there is an arbitrage opportunity, then it corresponds to a point in  $L$  such that each coordinate is non-negative and at least one coordinate is strictly positive. Suppose there is a point in  $L$  such that each coordinate is non-negative. Then this is given by  $(x + y, x - y, -y, -x + y)$  where

$$x + y \geq 0, \quad (2)$$

$$x - y \geq 0, \quad (3)$$

$$-y \geq 0, \quad (4)$$

$$-x + y \geq 0. \quad (5)$$

Together, (2) and (4) imply  $x \geq 0$ , but (4) and (5) imply  $x \leq 0$ , so  $x = 0$ . Also (4) implies  $y \leq 0$ , but (2) and (5) imply  $y \geq 0$ . Therefore, if there is a point in  $L$  such that each coordinate is non-negative, then that point must be the origin in  $\mathbb{R}^4$ . It follows that the market is viable and there are no arbitrage opportunities.

- (c) Suppose  $P^*$  is an EMM such that  $P(\{\omega_i\}) = p_i$  for  $i = 1, 2, 3, 4$ . Then  $p_i > 0$  for each  $i$  and  $p_1 + p_2 + p_3 + p_4 = 1$ . Let  $\vec{p} = (p_1, p_2, p_3, p_4)$  be the vector associated with the probabilities  $p_i$ . Then the martingale property can be presented as follows

$$E^{P^*}[\Delta S_1^{*,j} | \mathcal{F}_0] = p_1 \Delta S_1^{*,j}(\omega_1) + \dots + p_4 \Delta S_1^{*,j}(\omega_4) = \vec{p} \cdot \Delta S_1^{*,j} = 0, \quad j = 1, 2,$$

where  $\cdot$  represents the dot product. Therefore if  $P^*$ , with probabilities vector  $\vec{p}$ , is an EMM, it satisfies

$$\vec{p} \cdot \Delta S_1^{*,j} = 0, \quad j = 1, 2,$$

$$\vec{p} \cdot \vec{1} = 1,$$

$$0 < \vec{p} < 1.$$

Above we have used the notation that  $\vec{1} = (1, 1, 1, 1)$  and  $0 < \vec{p} < 1$  means that each coordinate  $p_i$  satisfies  $0 < p_i < 1$ . Plugging into the above system, we get

$$p_1 + p_2 - p_4 = 0,$$

$$p_1 - p_2 - p_3 + p_4 = 0,$$

$$p_1 + p_2 + p_3 + p_4 = 1,$$

$$0 < p_1, p_2, p_3, p_4 < 1.$$

Solving, we find that  $P^*$  is an EMM if and only if  $\vec{p}$  satisfies

$$\vec{p} = (p_1, p_2, p_3, p_4) \in \{(x, 1/2 - 2x, 2x, 1/2 - x) : x \in (0, 1/4)\}.$$

□