

Chapter 4

Black-Scholes Model

In this chapter we consider a simple continuous (in both time and space) financial market model called the Black-Scholes model. This can be viewed as a continuous analogue of the binomial model. Indeed, it can be obtained as a limit of a sequence of binomial models (with appropriate parameters).

Here $\log$ denotes the natural logarithm.

4.1 Black-Scholes Model

We consider a finite time interval $[0, T]$ for some $T < \infty$ as the interval during which trading may take place.

All of the random variables in this chapter will be defined on the common canonical space $\Omega = \{\omega : [0, T] \rightarrow \mathbb{R}, \omega \text{ is continuous}\}$ (we choose this space because it is the canonical path space for one-dimensional Brownian motion, which is the source of randomness in our Black-Scholes model). Let $\mathcal{F}^o = \sigma\{\omega(t) : 0 \leq t \leq T\}$ and P denote the probability measure on $(\Omega, \mathcal{F}^o)$ such that the canonical process W defined by

$$W_t(\omega) = \omega(t), \quad \omega \in \Omega, \quad t \in [0, T] \quad (4.1)$$

is a standard one-dimensional Brownian motion under P , i.e., P is standard Wiener measure. Let $(\Omega, \mathcal{F}, P)$ denote the completion of $(\Omega, \mathcal{F}^o, P)$ and define the filtration $\{\mathcal{F}_t, t \in [0, T]\}$ such that for each t , $\mathcal{F}_t = \sigma\{W_s : 0 \leq s \leq t\}^\sim$, the smallest σ -algebra with respect to which W_s is measurable for each $s \in [0, t]$ and which contains all of the P -null sets in $\mathcal{F}$ (as usual, the superscript $\sim$ denotes augmentation with the P -null sets). Note that $\mathcal{F} = \mathcal{F}_T$. Expectations with respect to P will be denoted by $E[\cdot]$, unless there is more than one probability measure under consideration, in which case we shall use E^P in place of E . A (stochastic) process will be a collection of real-valued random variables $\{Z_t, t \in$

$[0, T]$. Such a process is adapted if Z_t is $\mathcal{F}_t$ -measurable for each $t \in [0, T]$. The process is (right) continuous if its paths are (right) continuous. Two stochastic processes $\{Y_t, t \in [0, T]\}$ and $\{Z_t, t \in [0, T]\}$ defined on $(\Omega, \mathcal{F}, P)$ are

- (i) *versions of one another* if $P(Y_t = Z_t) = 1$ for all $t \in [0, T]$,
- (ii) *indistinguishable* if $P(Y_t = Z_t \text{ for all } t \in [0, T]) = 1$.

(Note that for (ii) it is implicit that the event $\{Y_t = Z_t \text{ for all } t \in [0, T]\}$ is measurable.) Two right continuous processes that are versions of one another are indistinguishable. Two processes will be regarded as being equal if they are indistinguishable.

Our Black-Scholes model has two assets, a (risky) stock with price process $S = \{S_t, t \in [0, T]\}$ and a (riskless) bond with price process $B = \{B_t, t \in [0, T]\}$. These processes are given by

$$S_t = S_0 \exp \left(\left(\mu - \frac{1}{2} \sigma^2 \right) t + \sigma W_t \right), \quad t \in [0, T], \quad (4.2)$$

$$B_t = e^{rt} \quad t \in [0, T], \quad (4.3)$$

where $r > 0$, $\mu \in \mathbb{R}$, $\sigma > 0$, and $S_0 > 0$ is a positive constant. The parameter σ is called the *volatility*. It is well known (cf. Chung and Williams [2], Theorem 6.2) that $\{e^{-\mu t} S_t, \mathcal{F}_t, t \in [0, T]\}$ is an L^2 -martingale under P . The processes S, B are continuous, adapted, and satisfy the following dynamic equations under P :

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \quad t \in [0, T], \quad (4.4)$$

$$dB_t = r B_t dt, \quad t \in [0, T], \quad (4.5)$$

Remark. To see the connection with the binomial model, note that in the Black-Scholes model, $\log(S_t/S_0) = (\mu - \frac{1}{2}\sigma^2)t + \sigma W_t$, a Brownian motion with drift, whereas in the binomial model, $\log(S_t/S_0) = \sum_{i=1}^t \xi_i$, which is a (possibly biased) random walk. It is well known that a random walk with appropriate rescaling and parameter values can be approximated by a Brownian motion with drift.

A *trading strategy* is a two-dimensional stochastic process $\phi = \{\phi_t = (\alpha_t, \beta_t), t \in [0, T]\}$ satisfying

- (i) $\phi : [0, T] \times \Omega \rightarrow \mathbb{R}^2$ is $(\mathcal{B}_T \times \mathcal{F}_T)$ -measurable, where $\phi(t, \omega) = \phi_t(\omega)$,
- (ii) ϕ is adapted, i.e., $\phi_t \in \mathcal{F}_t$ for each $t \in [0, T]$,
- (iii) $\int_0^T \alpha_t^2 dt < \infty$ and $\int_0^T |\beta_t| dt < \infty$ almost surely.

These conditions ensure that integrals of the form $\int_0^t \alpha_s dS_s$ and $\int_0^t \beta_s dB_s$ are finite P -a.s. and when considered as functions of t define adapted stochastic processes. The *value* at time t of the portfolio associated with ϕ is given by

$$V_t(\phi) = \alpha_t S_t + \beta_t B_t. \quad (4.6)$$

A trading strategy ϕ is said to be *self-financing* if P -a.s.,

$$V_t(\phi) = V_0(\phi) + \int_0^t \alpha_s dS_s + \int_0^t \beta_s dB_s, \quad t \in [0, T], \quad (4.7)$$

or, in other words,

$$dV_t(\phi) = \alpha_t dS_t + \beta_t dB_t, \quad t \in [0, T]. \quad (4.8)$$

Thus, changes in the value of the portfolio result only from changes in the values of the assets, so that there is no external infusion of capital and no spending of wealth.

Let ϕ be a trading strategy. The discounted stock price process and discounted value process associated with ϕ are given by

$$S_t^* = \frac{S_t}{B_t} = e^{-rt} S_t, \quad t \in [0, T], \quad (4.9)$$

$$V_t^*(\phi) = \frac{V_t(\phi)}{B_t} = e^{-rt} V_t(\phi), \quad t \in [0, T], \quad (4.10)$$

respectively. In particular, by (4.4) we have under P :

$$dS_t^* = -rS_t^* dt + e^{-rt} dS_t = (\mu - r)S_t^* dt + \sigma S_t^* dW_t, \quad t \in [0, T]. \quad (4.11)$$

Lemma 4.1.1 *A trading strategy ϕ is self-financing if and only if P -a.s.,*

$$V_t^*(\phi) = V_0(\phi) + \int_0^t \alpha_s dS_s^*, \quad \text{for all } t \in [0, T]. \quad (4.12)$$

PROOF. In the following, for convenience, we use the differential forms of integral equations which hold P -a.s. However, each of the steps could be written in integral form, to give a totally rigorous proof. The essential aspect is that we use the rules of stochastic calculus for manipulating differentials at each stage.

Suppose that ϕ is a self-financing trading strategy. Then, by (4.10), (4.6), and (4.8), we have for each $t \in [0, T]$,

$$\begin{aligned} dV_t^*(\phi) &= -re^{-rt} V_t(\phi) dt + e^{-rt} dV_t(\phi) \\ &= e^{-rt} (-r\alpha_t S_t dt - r\beta_t B_t dt + \alpha_t dS_t + \beta_t dB_t). \end{aligned}$$

Using (4.4), (4.5), and (4.11), the above yields

$$dV_t^*(\phi) = e^{-rt} \alpha_t ((\mu - r)S_t dt + \sigma S_t dW_t) = \alpha_t dS_t^*, \quad (4.13)$$

for each $t \in [0, T]$, which proves (4.12).

Conversely, suppose that (4.12) holds. Then using (4.11), (4.5), and (4.6), for each $t \in [0, T]$ we have

$$\begin{aligned} dV_t^*(\phi) &= \alpha_t dS_t^* = e^{-rt} \alpha_t (-rS_t dt + dS_t) \\ &= e^{-rt} (-r\alpha_t S_t dt - r\beta_t B_t dt + \beta_t dB_t + \alpha_t dS_t) \\ &= e^{-rt} (-rV_t(\phi) dt + \alpha_t dS_t + \beta_t dB_t). \end{aligned}$$

Then,

$$\begin{aligned} dV_t(\phi) &= d(e^{rt} V_t^*(\phi)) = r e^{rt} V_t^*(\phi) dt + e^{rt} dV_t^*(\phi) \\ &= r V_t(\phi) dt + (-rV_t(\phi) dt + \alpha_t dS_t + \beta_t dB_t) \\ &= \alpha_t dS_t + \beta_t dB_t, \end{aligned}$$

and hence ϕ is self-financing. $\square$

A self-financing trading strategy ϕ is an *arbitrage opportunity* if

$$V_0(\phi) = 0, \quad V_T(\phi) \geq 0, \quad \text{and} \quad E[V_T(\phi)] > 0. \quad (4.14)$$

In contrast to the discrete time case, where we had only finitely many trading times and gains or losses are controlled by the initial investment, in continuous time, one can change one's strategy infinitely many times in a finite time interval and thereby obtain unbounded gains or losses by use of a doubling type of strategy. The following is a concrete example of such a strategy.

Example. Consider the Black-Scholes model with $r = 0$, $\mu = 0$, and $\sigma = 1$. Then

$$B_t = 1 \quad \text{and} \quad dS_t = S_t dW_t, \quad t \in [0, T]. \quad (4.15)$$

Define

$$I(t) = \int_0^t \frac{1}{\sqrt{T-s}} dW_s, \quad \text{for all } 0 \leq t < T. \quad (4.16)$$

Then the quadratic variation of I is given by

$$[I]_t = \int_0^t \frac{1}{T-s} ds = \log \left(\frac{T}{T-t} \right), \quad \text{for each } t < T. \quad (4.17)$$

Note that $[I]_0 = 0$, $[I]_t$ is increasing with t , and $[I]_t \rightarrow \infty$ as $t \rightarrow T$. It follows that I can be time changed to a Brownian motion (cf. Chung-Williams [2], Section 9.3) and in particular, for each $a > 0$ and

$$\tau_a \equiv \inf\{t \in [0, T] : I(t) = a\} \wedge T, \quad (4.18)$$

we have $0 < \tau_a < T$ P -a.s. Fix $a > 0$ and let $\phi = \{(\alpha_t, \beta_t), t \in [0, T]\}$, where for each $t \in [0, T]$,

$$\alpha_t = (T - t)^{-\frac{1}{2}} S_t^{-1} 1_{\{t \leq \tau_a\}}, \quad (4.19)$$

$$\beta_t = I(t \wedge \tau_a) - \alpha_t S_t. \quad (4.20)$$

Then ϕ is a trading strategy with value process

$$V_t(\phi) = I(t \wedge \tau_a) = \int_0^{t \wedge \tau_a} \frac{1}{\sqrt{T-s}} dW_s \quad (4.21)$$

$$= \int_0^t \alpha_s S_s dW_s = \int_0^t \alpha_s dS_s, \quad (4.22)$$

where we have used the fact that S solves (4.4) with $\mu = 0$ and $\sigma = 1$. Since $r = 0$, it follows that (4.12) holds and hence ϕ is self-financing. Note that $V_0(\phi) = 0$ and $V_T(\phi) = I(\tau_a) = a > 0$ and so ϕ is an arbitrage opportunity.

There are several ways to add conditions to ϕ to rule out such arbitrage strategies. These usually amount to constraints on the size of integrals of ϕ with respect to $d(S, B)$. Since pricing of contingent claims based on this model will use an equivalent martingale measure (or risk neutral probability), these conditions are often phrased in terms of such a probability measure. Some authors require that the associated value process is L^2 -bounded under an equivalent martingale measure, some require that the discounted value process is a martingale under such a measure, and some put integrability constraints directly on ϕ . Before discussing our conditions on ϕ , we introduce the equivalent martingale measure.

4.2 Equivalent Martingale Measure

Definition 4.2.1 *Two probability measures Q and $\tilde{Q}$ on $(\Omega, \mathcal{F})$ are equivalent (or mutually absolutely continuous) if*

$$Q(A) = 0 \text{ is equivalent to } \tilde{Q}(A) = 0 \text{ for each } A \in \mathcal{F}. \quad (4.23)$$

An equivalent martingale measure, also called a risk neutral probability, is a probability measure P^ on $(\Omega, \mathcal{F})$ such that P^* is equivalent to P and $\{S_t, \mathcal{F}_t, t \in [0, T]\}$ is a martingale under P^* .*

We wish to show that such a P^* exists. For this note that

$$S_t^* = S_0 \exp \left(\left(\mu - r - \frac{1}{2} \sigma^2 \right) t + \sigma W_t \right) \quad (4.24)$$

and in particular, P -a.s.,

$$dS_t^* = (\mu - r) S_t^* dt + \sigma S_t^* dW_t, \quad t \in [0, T]. \quad (4.25)$$

The second term on the right in (4.25), when integrated, defines a martingale under P . We seek a probability measure P^* equivalent to P such that the discounted stock price process S^* satisfies a differential equation like (4.25), but without the drift term $(\mu - r)S_t^* dt$. For this, we use a simple form of the Girsanov transformation for changing the drift of a Brownian motion. Let

$$\theta = \frac{\mu - r}{\sigma},$$

$$\Lambda_t = \exp\left(-\theta W_t - \frac{1}{2}\theta^2 t\right), \quad t \in [0, T].$$

Then it is well known (cf. [2], Theorem 6.2) that $\{\Lambda_t, \mathcal{F}_t, t \in [0, T]\}$ is a positive martingale under P . On $(\Omega, \mathcal{F})$, we define a new probability measure P^* so that

$$\frac{dP^*}{dP} = \Lambda_T \quad \text{on } \mathcal{F}, \quad (4.26)$$

i.e.,

$$P^*(A) = E^P[1_A \Lambda_T], \quad A \in \mathcal{F}, \quad (4.27)$$

where we have added the superscript P to the expectation E to emphasize that it is computed under the probability measure P . Similarly, we shall use E^{P^*} to denote the expectation operator under P^* . Note that, since $\{\Lambda_t, \mathcal{F}_t, t \in [0, T]\}$ is a P -martingale, $P^*(\Omega) = E^P[\Lambda_T] = \Lambda_0 = 1$ and so P^* is indeed a probability measure on $(\Omega, \mathcal{F})$. Since $\Lambda_T > 0$, it follows that for each $A \in \mathcal{F}$, $P^*(A) = 0$ if and only if $P(A) = 0$. Thus, P^* is equivalent to P . All that remains is to show that the discounted stock price process S^* is a martingale under P^* . For this, let

$$\tilde{W}_t = W_t + \theta t, \quad t \in [0, T]. \quad (4.28)$$

By Girsanov's theorem (cf. [2], Section 9.4), $\{\tilde{W}_t, \mathcal{F}_t, t \in [0, T]\}$ is a standard Brownian motion under P^* . Thus, for each $t \in [0, T]$, by (4.24) we have

$$S_t^* = S_0 \exp\left(\left(\mu - r - \frac{1}{2}\sigma^2\right)t + \sigma(\tilde{W}_t - \theta t)\right) \quad (4.29)$$

$$= S_0 \exp\left(-\frac{1}{2}\sigma^2 t + \sigma \tilde{W}_t\right), \quad (4.30)$$

and so P^* -a.s.,

$$dS_t^* = \sigma S_t^* d\tilde{W}_t, \quad t \in [0, T], \quad (4.31)$$

where $\tilde{W}$ is a standard Brownian motion under P^* . It is well known that the form (4.30) is that of an L^2 -martingale with respect to $\{\mathcal{F}_t\}$, under P^* (cf. [2], Theorem 6.2). Thus, P^* is an equivalent martingale measure.

An *admissible strategy* is a self-financing trading strategy ϕ such that $\{V_t^*(\phi), \mathcal{F}_t, t \in [0, T]\}$ is a martingale under the equivalent martingale measure P^* .

4.3 European Contingent Claims

A *European contingent claim* is represented by an $\mathcal{F}_T$ -measurable random variable X . A *replicating strategy* for a European contingent claim $X \in \mathcal{F}_T$ is an admissible strategy ϕ such that $V_T(\phi) = X$.

We shall use the following form of the martingale representation theorem for Brownian motion to obtain a replicating strategy for suitably integrable X . For this theorem, it is important that we are using the filtration generated by the Brownian motion W (cf. Section 4.1).

A random variable $\tau : \Omega \rightarrow [0, T] \cup \{+\infty\}$ is a *stopping time* if

$$\{\omega \in \Omega : \tau(\omega) \leq t\} \in \mathcal{F}_t \quad \text{for each } t \in [0, T]. \quad (4.32)$$

A real-valued process $M = \{M_t, t \in [0, T]\}$ is a *local martingale* under P^* if it is right continuous, adapted, and there exists an increasing sequence of stopping times $\{\tau_n : n = 1, 2, \dots\}$ taking values in $[0, T] \cup \{+\infty\}$ such that

$$P^*(\tau_n \geq T \text{ for all sufficiently large } n) = 1, \quad (4.33)$$

and for each n , $M^n = \{M_{t \wedge \tau_n}, \mathcal{F}_t, t \in [0, T]\}$ is a martingale under P^* .

Theorem 4.3.1 *Suppose that $M = \{M_t, \mathcal{F}_t, t \in [0, T]\}$ is a local martingale under P^* . Then there exists a $(\mathcal{B}_T \times \mathcal{F}_T)$ -measurable, adapted process $\eta = \{\eta_t : t \in [0, T]\}$ such that P^* -a.s., $\int_0^T \eta_s^2 ds < \infty$ and P^* -a.s.,*

$$M_t = M_0 + \int_0^t \eta_s d\tilde{W}_s, \quad \text{for all } t \in [0, T]. \quad (4.34)$$

In particular, M has a continuous version.

PROOF. For a proof of this martingale representation theorem, see Revuz and Yor [4], Theorem V.3.5. $\square$

Remark. Note that P and P^* have the same null sets and so an equality holds P -a.s. if and only if it holds P^* -a.s. By (4.28) and the equivalence of P^* and P , we have that $\mathcal{F}_t = \sigma\{\tilde{W}_s, s \in [0, t]\}^\sim$ for each $t \in [0, T]$, where the superscript $\sim$ denotes augmentation by the P^* -null sets.

Theorem 4.3.2 *Suppose that X is an $\mathcal{F}_T$ -measurable random variable such that $E^{P^*}[|X|] < \infty$. Then there exists a replicating strategy ϕ for X . Moreover, for $X^* = X/B_T$, we have that for each $t \in [0, T]$, P^* -a.s.,*

$$V_t^*(\phi) = E^{P^*}[X^* | \mathcal{F}_t]. \quad (4.35)$$

PROOF. We first prove that (4.35) holds if ϕ is a replicating strategy for X . Indeed, for such a ϕ , we have for fixed $t \in [0, T]$ that P^* -a.s.,

$$V_t^*(\phi) = E^{P^*} [V_T^*(\phi) \mid \mathcal{F}_t] = E^{P^*} [X^* \mid \mathcal{F}_t], \quad (4.36)$$

where the first equality follows from the admissibility of ϕ which implies that $\{V_t^*(\phi), \mathcal{F}_t, t \in [0, T]\}$ is a martingale under P^* , and the second equality follows from the replicating property of ϕ : $V_T(\phi) = X$.

In light of the above, to prove the existence of a replicating strategy ϕ , it is natural to consider

$$M_t = E^{P^*} [X^* \mid \mathcal{F}_t], \quad t \in [0, T]. \quad (4.37)$$

Note that $\{M_t, \mathcal{F}_t, t \in [0, T]\}$ is a P^* -martingale and hence it has a right continuous version. In fact, since the filtration is generated by Brownian motion, M has a continuous version (cf. Revuz and Yor [4], Theorems II.2.9, V.3.5). We again denote such a continuous version by M . Thus, M is a local martingale (take $\tau_n = +\infty$ for all n in the definition) and so by Theorem 4.3.1, there is an adapted process $\eta = \{\eta_t, t \in [0, T]\}$ satisfying the conditions in Theorem 4.3.1 and such that for each $t \in [0, T]$, P^* -a.s.,

$$M_t = M_0 + \int_0^t \eta_s d\tilde{W}_s. \quad (4.38)$$

Define

$$\alpha_t = \frac{\eta_t}{\sigma S_t^*}, \quad t \in [0, T]. \quad (4.39)$$

Then $\alpha = \{\alpha_t, t \in [0, T]\}$ is $(\mathcal{B}_T \times \mathcal{F}_T)$ -measurable, adapted and

$$\int_0^T \alpha_t^2 dt \leq \left(\sigma \inf_{t \in [0, T]} S_t^* \right)^{-2} \int_0^T \eta_t^2 dt. \quad (4.40)$$

Note that $S_t^* > 0$ for each $t \in [0, T]$, and S_t^* is a continuous function of $t \in [0, T]$. Therefore, $\inf_{t \in [0, T]} S_t^* > 0$. So it follows from the integrability property of η that P^* -a.s.,

$$\int_0^T \alpha_t^2 dt < \infty. \quad (4.41)$$

Define

$$\beta_t = M_t - \alpha_t S_t^*, \quad t \in [0, T]. \quad (4.42)$$

Then $\beta = \{\beta_t, t \in [0, T]\}$ is $(\mathcal{B}_T \times \mathcal{F}_T)$ -measurable, adapted and using Cauchy's inequality we have

$$\int_0^T |\beta_t| dt \leq \int_0^T |M_t| dt + \int_0^T |\alpha_t| S_t^* dt \quad (4.43)$$

$$\leq T \max_{t \in [0, T]} |M_t| + \left(\max_{t \in [0, T]} S_t^* \right) \left(\int_0^T \alpha_t^2 dt \right)^{\frac{1}{2}} T^{\frac{1}{2}}. \quad (4.44)$$

The last line above is finite P^* -a.s., by (4.41) and the fact that M and S^* are continuous processes. Let $\phi = \{(\alpha_t, \beta_t) : t \in [0, T]\}$. Then,

$$V_t^*(\phi) = \alpha_t S_t^* + \beta_t = M_t, \quad t \in [0, T]. \quad (4.45)$$

This, together with (4.34), the definition of α , and (4.31), gives P^* -a.s.,

$$V_t^*(\phi) = V_0(\phi) + \int_0^t \eta_s d\tilde{W}_s \quad (4.46)$$

$$= V_0(\phi) + \int_0^t \alpha_s \sigma S_s^* d\tilde{W}_s \quad (4.47)$$

$$= V_0(\phi) + \int_0^t \alpha_s dS_s^*, \quad (4.48)$$

for each $t \in [0, T]$. Therefore, since P and P^* have the same null sets, ϕ is a self-financing trading strategy. Combining this with (4.45) and the martingale property of M , we see that ϕ is admissible. Finally, taking $t = T$ in (4.45) implies that ϕ replicates X since $M_T = X^*$. $\square$

4.4 Arbitrage Free Price Process

Consider a European contingent claim $X \in \mathcal{F}_T$ such that $E^{P^*}[|X|] < \infty$. To determine the arbitrage free price process for this claim, we need a notion of admissible strategy for trading in stock, bond and the contingent claim.

We assume that any given price process $C = \{C_t, t \in [0, T]\}$ for the European contingent claim is a right continuous, adapted process, where C_t represents the price of the European contingent claim at time t . (Adaptedness is the minimal assumption we would expect on this process and right continuity is a reasonable regularity assumption since all semimartingales have right continuous versions.) We suppose that for a given price process C , there is a set of admissible strategies Ψ_C for trading in stock, bond and the contingent claim. At a minimum, an element ψ of Ψ_C should be a three-dimensional adapted process $\psi = \{\psi_t = (\alpha_t, \beta_t, \gamma_t), t \in [0, T]\}$.

To show uniqueness of the price process, we do not want to place unnecessary restrictions on C just so that it can be used as an integrator (in defining self-financing strategies). In fact, to show uniqueness, we shall only need the reasonable assumption that rather simple strategies of the following type are admissible: those where one buys or sells one contingent claim, or does nothing, at some deterministic time t^* , and then holds that position in the claim until time T , but where one may trade as usual in the stock and bond over $[t^*, T]$. More precisely, we assume that strategies of the following form are in Ψ_C :

$$\psi_t(\omega) = 1_{\{t \geq t^*\}} 1_A(\omega) (\alpha_t(\omega), \beta_t(\omega) + \delta_t(\omega), \gamma_t(\omega)) \quad \omega \in \Omega, \quad t \in [0, T], \quad (4.49)$$

are admissible, where t^* is a fixed time in $[0, T]$, $A \in \mathcal{F}_{t^*}$, $\{(\alpha_t, \beta_t), t \in [0, T]\}$ is an admissible trading strategy in the primary (stock-bond) market, $\gamma_t = +1$ for all t or $\gamma_t = -1$ for all t , $\delta_t = \delta_{t^*}$ for all t is chosen such that $\alpha_{t^*}S_{t^*} + (\beta_{t^*} + \delta_{t^*})B_{t^*} + \gamma_{t^*}C_{t^*} = 0$. Under such a strategy, the investor does nothing prior to the time t^* . On A^c , the investor continues to do nothing for the remaining interval $[t^*, T]$. On A , if $\gamma_t \equiv +1$, at t^* the investor buys one contingent claim for C_{t^*} and invests $-C_{t^*}$ in the stock-bond market according to $(\alpha_t, \beta_t + \delta_t)$ for $t \in [t^*, T]$. On A , if $\gamma_t \equiv -1$, instead of buying one contingent claim at time t^* , the investor sells one contingent claim at t^* and invests the proceeds C_{t^*} according to $(\alpha_t, \beta_t + \delta_t)$ for $t \in [t^*, T]$. Since the value of the strategy is zero at time t^* and δ_t and γ_t do not vary with $t \geq t^*$, the strategy will be naturally self-financing, since (α, β) is assumed to have this property.

We will prove below that when the price process C is a continuous version of

$$e^{rt}E^{P^*}[X^* | \mathcal{F}_t], \quad t \in [0, T], \quad (4.50)$$

it is an arbitrage free price process. For this, we need similar constraints on the admissible strategies to what we had in the case of the primary market, to ensure that doubling type strategies are ruled out. Note that in this case, if $\phi^* = (\alpha^*, \beta^*)$ is a replicating strategy for X , then $C_t^* = e^{-rt}C_t = V_t^*(\phi^*)$ defines a martingale under P^* and P^* -a.s.,

$$dC_t^* = dV_t^*(\phi^*) = \alpha_t^*dS_t^* = \sigma\alpha_t^*S_t^*d\tilde{W}_t. \quad (4.51)$$

Consequently, in this case, we let the class of admissible strategies Ψ_C be the set of three-dimensional processes $\psi = (\alpha, \beta, \gamma)$ such that

- (i) $\psi : [0, T] \times \Omega \rightarrow \mathbb{R}^3$ is $(\mathcal{B}_T \times \mathcal{F}_T)$ -measurable, where $\psi(t, \omega) = \psi_t(\omega)$,
- (ii) ψ is adapted, i.e., $\psi_t \in \mathcal{F}_t$ for each $t \in [0, T]$,
- (iii) $\int_0^T \alpha_t^2 dt < \infty$, $\int_0^T |\beta_t| dt < \infty$ almost surely, $\int_0^T \gamma_t^2 (\alpha_t^*)^2 dt < \infty$,
- (iv)

$$V_t(\psi) \equiv \alpha_t S_t + \beta_t B_t + \gamma_t C_t \quad (4.52)$$

$$= V_0(\psi) + \int_0^t \alpha_s dS_s + \int_0^t \beta_s dB_s + \int_0^t \gamma_s dC_s, \quad (4.53)$$

- (v) $\{V_t^*(\psi) = e^{-rt}V_t(\psi), \mathcal{F}_t, t \in [0, T]\}$ is a martingale under P^* .

Properties (i)–(iii) above ensure that the stochastic integrals appearing in (4.53) are well defined, property (iv) is the self-financing property of ψ , and (v) amounts to a constraint on the value of ψ to rule out doubling strategies. Note that since S^* and C^* are already local martingales under P^* , this last condition amounts to some control on the size of ψ .

Given a price process C for the contingent claim, an *arbitrage opportunity* in the stock-bond-contingent claim market is an admissible strategy $\psi \in \Psi_C$ such that $V_0(\psi) = 0$, $V_T(\psi) > 0$ and $E^{P^*}[V_T(\psi)] > 0$.

Theorem 4.4.1 *The arbitrage free price process for the P^* -integrable European contingent claim X is given by $C_t = V_t(\phi^*) = e^{-rt}V_t^*(\phi^*)$, $t \in [0, T]$, where ϕ^* is a replicating strategy for X and $\{V_t^*(\phi^*), t \in [0, T]\}$ is a continuous version of the martingale $E^{P^*}[X^* | \mathcal{F}_t]$, $t \in [0, T]$.*

PROOF. Consider a price process $\{C_t, t \in [0, T]\}$ for the European contingent claim. Suppose that $P^*(C_t \neq V_t(\phi^*))$ for some $t \in [0, T]$ is > 0 . Since the process C is assumed to be right continuous and $V(\phi^*)$ is continuous, there exists a $t^* \in [0, T]$ such that $P^*(C_{t^*} \neq V_{t^*}(\phi^*)) > 0$. Let $A = \{\omega : C_{t^*}(\omega) > V_{t^*}(\phi^*)(\omega)\}$ and $\tilde{A} = \{\omega : C_{t^*}(\omega) < V_{t^*}(\phi^*)(\omega)\}$. Then either $P^*(A) > 0$ or $P^*(\tilde{A}) > 0$. First suppose that $P^*(A) > 0$ and suppose that the replicating strategy has the form $\phi^* = (\alpha^*, \beta^*)$. Define $\psi = \{(\alpha_t, \beta_t, \gamma_t), t \in [0, T]\}$ by

$$\psi_t(\omega) = \begin{cases} (0, 0, 0), & t < t^*, \\ (0, 0, 0), & t \geq t^*, \omega \in A^c, \\ \left(\alpha_t^*(\omega), \beta_t^*(\omega) + \frac{C_{t^*}(\omega) - V_{t^*}(\phi^*)(\omega)}{B_{t^*}}, -1\right), & t \geq t^*, \omega \in A, \end{cases} \quad (4.54)$$

or in other words, for $t \in [0, T]$, $\omega \in \Omega$,

$$\psi_t(\omega) = 1_{\{t \geq t^*\}} 1_A(\omega) \left(\alpha_t^*, \beta_t^* + \frac{C_{t^*} - V_{t^*}(\phi^*)}{B_{t^*}}, -1 \right)(\omega). \quad (4.55)$$

Clearly, the value of the portfolio represented by ψ_t is zero for $t \leq t^*$, and so the value at time zero is zero. Moreover, the value at time T is zero on A^c . Also, since ϕ^* is a replicating strategy for X , on A , the value at time T is $\left(\frac{C_{t^*} - V_{t^*}(\phi^*)}{B_{t^*}}\right) B_T$, which is strictly positive. Since $P^*(A) > 0$, ψ is an arbitrage opportunity.

Similarly, if $P^*(\tilde{A}) > 0$, then ψ defined for $t \in [0, T]$, $\omega \in \Omega$ by

$$\psi_t(\omega) = 1_{\{t \geq t^*\}} 1_{\tilde{A}}(\omega) \left(-\alpha_t^*, -\beta_t^* + \frac{V_{t^*}(\phi^*) - C_{t^*}}{B_{t^*}}, 1 \right)(\omega), \quad (4.56)$$

is an arbitrage opportunity.

Thus, we have shown that the only possible arbitrage free price process (up to indistinguishability) is given by $C_t = V_t(\phi^*)$, $t \in [0, T]$. Next we show that this price process is arbitrage free. For a contradiction, suppose that with this price process, there is $\psi \in \Psi_C$ such that $V_0(\psi) = 0$, $V_T(\psi) \geq 0$ and $E^{P^*}[V_T(\psi)] > 0$. Now, by the admissibility assumption on ψ , the discounted value process $V^*(\psi)$ is a martingale under P^* and so

$$0 = V_0(\psi) = E^{P^*}[V_T^*(\psi)] = e^{-rT} E^{P^*}[V_T(\psi)] > 0, \quad (4.57)$$

a contradiction. Thus, there cannot be an arbitrage opportunity with this price process. $\square$

4.5 European Call Options

In this section, we apply the results of the previous section to compute the arbitrage free price process for a European call option and we also identify a replicating strategy for the option.

Let $X = (S_T - K)^+$ be a European call option with strike price K and expiration time T . Notice that $X \leq S_T$ and $E^{P^*}[S_T] = S_0 e^{rt} < \infty$. Hence, by Theorem 4.4.1, the arbitrage free price process $\{C_t, t \in [0, T]\}$ is a continuous adapted process such that for each $t \in [0, T]$, if $C_t^* = C_t e^{-rt}$, then we have P^* -a.s.,

$$C_t^* = E^{P^*}[X^* | \mathcal{F}_t] = E^{P^*}[(S_T^* - K^*)^+ | \mathcal{F}_t], \quad (4.58)$$

where $K^* = e^{-rT}K$. It follows from (4.30) that P^* -a.s., for each $t \in [0, T]$,

$$S_T^* = S_t^* \frac{S_0 \exp\left(\sigma \tilde{W}_T - \frac{1}{2}\sigma^2 T\right)}{S_0 \exp\left(\sigma \tilde{W}_t - \frac{1}{2}\sigma^2 t\right)} = S_t^* \exp\left(\sigma (\tilde{W}_T - \tilde{W}_t) - \frac{1}{2}\sigma^2(T-t)\right). \quad (4.59)$$

On substituting this into the right member of (4.58), we obtain that for $t \in [0, T]$, P^* -a.s.,

$$C_t^* = E^{P^*}\left[\left(S_t^* \exp\left(\sigma (\tilde{W}_T - \tilde{W}_t) - \frac{1}{2}\sigma^2(T-t)\right) - K^*\right)^+ | \mathcal{F}_t\right]. \quad (4.60)$$

Using the fact that $S_t^* \in \mathcal{F}_t$ and that $\tilde{W}_T - \tilde{W}_t$ is a normal random variable with mean zero and variance $T-t$ that is independent of $\mathcal{F}_t$ under P^* (since $\mathcal{F}_t$ is generated by $\tilde{W}_{\cdot \wedge t}$ and the P^* -null sets), it follows that for $t \in [0, T]$, P^* -a.s.,

$$C_t^* = \frac{1}{\sqrt{2\pi(T-t)}} \int_{-\infty}^{\infty} \left(S_t^* \exp\left(\sigma y - \frac{1}{2}\sigma^2(T-t)\right) - K^*\right)^+ \exp\left(\frac{-y^2}{2(T-t)}\right) dy,$$

and $C_T^* = (S_T^* - K^*)^+$. Let $z = y/\sqrt{T-t}$. Then the expression on the right above becomes

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(S_t^* \exp\left(\sigma z \sqrt{T-t} - \frac{1}{2}\sigma^2(T-t)\right) - K^*\right)^+ \exp\left(-\frac{1}{2}z^2\right) dz. \quad (4.61)$$

Define $f : [0, T] \times (0, \infty)$ by $f(T, x) = (x - K^*)^+$ for all $x \in (0, \infty)$, and for $t \in [0, T)$, $x \in (0, \infty)$,

$$f(t, x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(x \exp \left(\sigma z \sqrt{T-t} - \frac{1}{2} \sigma^2 (T-t) \right) - K^* \right)^+ \exp \left(-\frac{1}{2} z^2 \right) dz. \quad (4.62)$$

Then, by the above, for each $t \in [0, T]$, P^* -a.s.,

$$C_t^* = f(t, S_t^*). \quad (4.63)$$

For $t \in [0, T)$ and $x \in (0, \infty)$, let $l_t(x)$ be such that the integrand in (4.62) is positive if and only if $z > l_t(x)$, i.e.,

$$l_t(x) = \frac{\log \left(\frac{K^*}{x} \right)}{\sigma \sqrt{T-t}} + \frac{1}{2} \sigma \sqrt{T-t}. \quad (4.64)$$

Thus, for $t \in [0, T)$ and $x \in (0, \infty)$,

$$\begin{aligned} f(t, x) &= \frac{x}{\sqrt{2\pi}} \int_{l_t(x)}^{\infty} \exp \left(\sigma z \sqrt{T-t} - \frac{1}{2} \sigma^2 (T-t) - \frac{z^2}{2} \right) dz \\ &\quad - \frac{K^*}{\sqrt{2\pi}} \int_{l_t(x)}^{\infty} \exp \left(-\frac{z^2}{2} \right) dz \\ &= \frac{x}{\sqrt{2\pi}} \int_{l_t(x)}^{\infty} \exp \left(\frac{-(z - \sigma \sqrt{T-t})^2}{2} \right) dz - K^* \Phi(-l_t(x)) \\ &= \frac{x}{\sqrt{2\pi}} \int_{l_t(x) - \sigma \sqrt{T-t}}^{\infty} \exp \left(-\frac{u^2}{2} \right) du - K^* \Phi(-l_t(x)) \\ &= x \Phi \left(-l_t(x) + \sigma \sqrt{T-t} \right) - K^* \Phi(-l_t(x)) \end{aligned}$$

where we have used the change of variable $u = z - \sigma \sqrt{T-t}$ and Φ is the cumulative distribution function for a standard normal random variable with mean zero and variance one:

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^z \exp \left(-\frac{u^2}{2} \right) du, \quad z \in (-\infty, \infty). \quad (4.65)$$

Thus, for $t \in [0, T)$, P^* -a.s.,

$$C_t^* = f(t, S_t^*) = S_t^* \Phi \left(\frac{\log \left(\frac{S_t^*}{K^*} \right)}{\sigma \sqrt{T-t}} + \frac{1}{2} \sigma \sqrt{T-t} \right) - K^* \Phi \left(\frac{\log \left(\frac{S_t^*}{K^*} \right)}{\sigma \sqrt{T-t}} - \frac{1}{2} \sigma \sqrt{T-t} \right), \quad (4.66)$$

and in particular, P^* -a.s.,

$$C_0 = C_0^* = S_0 \Phi \left(\frac{\log \left(\frac{S_0}{K^*} \right)}{\sigma \sqrt{T}} + \frac{1}{2} \sigma \sqrt{T} \right) - K^* \Phi \left(\frac{\log \left(\frac{S_0}{K^*} \right)}{\sigma \sqrt{T}} - \frac{1}{2} \sigma \sqrt{T} \right), \quad (4.67)$$

which is the famous *Black-Scholes formula*.

The Black-Scholes formula gives the arbitrage free price for the European call option, but it does not immediately give a replicating or hedging strategy for the option. Recall that the existence of such a hedging strategy was guaranteed by the martingale representation theorem (cf. the proof of Theorem 4.4.1), but that result is non-constructive. We will now use stochastic calculus and the function f defined above to obtain such a hedging strategy.

Note that $f : [0, T] \times (0, \infty)$ is defined for $t \in [0, T)$, $x \in (0, \infty)$ by

$$f(t, x) = x\Phi\left(\frac{\log\left(\frac{x}{K^*}\right)}{\sigma\sqrt{T-t}} + \frac{1}{2}\sigma\sqrt{T-t}\right) - K^*\Phi\left(\frac{\log\left(\frac{x}{K^*}\right)}{\sigma\sqrt{T-t}} - \frac{1}{2}\sigma\sqrt{T-t}\right),$$

and

$$f(T, x) = (x - K^*)^+. \quad (4.68)$$

It is straightforward to verify from the above formula that $f \in C^{1,2}([0, T] \times (0, \infty))$, i.e., as a function of $(t, x) \in [0, T] \times (0, \infty)$, $f(t, x)$ is once continuously differentiable in t and twice continuously differentiable in x . In particular, the first partial derivative with respect to x is given for $t \in [0, T)$, $x \in (0, \infty)$, by

$$f_x(t, x) = \Phi\left(\frac{\log\left(\frac{x}{K^*}\right)}{\sigma\sqrt{T-t}} + \frac{1}{2}\sigma\sqrt{T-t}\right), \quad (4.69)$$

which is a bounded continuous function. In addition, by using the alternative representation (cf. (4.60)):

$$f(t, x) = E^{P^*} \left[\left(x \exp \left(\sigma(\tilde{W}_T - \tilde{W}_t) - \frac{1}{2}\sigma^2(T-t) \right) - K^* \right)^+ \right], \quad (4.70)$$

for $(t, x) \in [0, T] \times (0, \infty)$, the uniform integrability of $\{\exp(\sigma(\tilde{W}_T - \tilde{W}_t) - \frac{1}{2}\sigma^2(T-t)), t \in [0, T]\}$, and dominated convergence, we can readily verify that f is continuous on $[0, T] \times (0, \infty)$. Note that by the continuity of f , we may and do assume that P^* -a.s., $C_t^* = f(t, S_t^*)$ for all $t \in [0, T]$, (i.e., the exceptional P^* -null set does not depend on t). We can apply Itô's formula to $f(t, S_t^*)$ for $t < T$ to obtain P^* -a.s., for all $t < T$,

$$f(t, S_t^*) = f(0, S_0) + \int_0^t f_s(s, S_s^*) ds + \int_0^t f_x(s, S_s^*) dS_s^* + \frac{1}{2} \int_0^t f_{xx}(s, S_s^*) d[S^*]_s, \quad (4.71)$$

where $f_s(s, x)$ denotes the first partial derivative of $f(s, x)$ with respect to s ; $f_x(s, x)$ and $f_{xx}(s, x)$ denote the first and second partial derivatives, respectively, of $f(s, x)$ with respect to x ; and $[S^*]$ denotes the quadratic variation process associated with S^* which satisfies $d[S^*]_s = \sigma^2(S_s^*)^2 ds$. The first and third

integrals above are path-by-path Riemann integrals and the second integral is a stochastic integral. Indeed, for each $t < T$, $\{\int_0^u f_x(s, S_s^*) dS_s^*, \mathcal{F}_u, u \in [0, t]\}$ is a continuous local martingale under P^* . On the other hand, $\{C_u^*, \mathcal{F}_u, u \in [0, t]\}$ is a continuous process and by (4.58) it is a martingale under P^* . Now, P^* -a.s., $C_u^* = f(u, S_u^*)$ for all $u \in [0, t]$, and so by (4.71) we have P^* -a.s. for all $u \in [0, t]$:

$$\int_0^u \left(f_s(s, S_s^*) + \frac{1}{2} \sigma^2 (S_s^*)^2 f_{xx}(s, S_s^*) \right) ds = C_u^* - C_0^* - \int_0^u f_x(s, S_s^*) dS_s^*. \quad (4.72)$$

Now, by the above discussion, the right member above is a continuous local martingale under P^* that starts from zero, and the left member is a continuous, adapted process whose paths are of bounded variation on $[0, t]$. It follows, by the uniqueness of the decomposition of a continuous semimartingale (cf. [2], Corollary 4.5), that P^* -a.s., both sides of (4.72) are zero for all $u \in [0, t]$. Since $t < T$ was arbitrary, this in fact holds true for all $u \in [0, T]$. In fact, not only is the left member zero P^* -a.s., but one can show (for example by direct verification using the formula for f) that f satisfies the *Black-Scholes partial differential equation*:

$$f_t + \frac{1}{2} \sigma^2 x^2 f_{xx} = 0 \quad \text{for all } (t, x) \in [0, T] \times (0, \infty).$$

(In fact, there are several different forms of the Black-Scholes partial differential equation, depending on whether one uses the function $f(t, x)$ for representing C^* or one uses the undiscounted function $e^{rt} f(t, x)$ for representing C .) Using the fact that P^* -a.s., the right member of (4.72) is zero for all $u \in [0, T]$, we have that P^* -a.s.,

$$C_t^* = C_0 + \int_0^t f_x(s, S_s^*) dS_s^* \quad \text{for all } t < T. \quad (4.73)$$

Comparing this with the self-financing property of trading strategies and keeping in mind that we want C^* to be the discounted value process for a replicating strategy, the above suggests that we define

$$\alpha_t = f_x(t, S_t^*) \quad \text{for each } t \in [0, T].$$

The definition of α_T is not critical (since it is only in force for an instant of time), as long as it is part of a self-financing strategy. However, we can define it in such a way that $\alpha_T = \lim_{t \rightarrow T} \alpha_t$ P^* -a.s. Note that since $P^*(S_T^* = K^*) = 0$ and S^* has continuous paths, we have that P^* -a.s.,

$$\lim_{t \rightarrow T} f_x(t, S_t^*) = \lim_{t \rightarrow T} \Phi \left(\frac{\log \left(\frac{S_t^*}{K^*} \right)}{\sigma \sqrt{T-t}} + \frac{1}{2} \sigma \sqrt{T-t} \right) \quad (4.74)$$

$$= 1_{\{S_T^* > K^*\}}. \quad (4.75)$$

Thus, we may and do define $\alpha_T = 1_{\{S_T^* > K^*\}}$. In order to maintain the appropriate value process, we define

$$\beta_t = C_t^* - \alpha_t S_t^*, \quad t \in [0, T].$$

It is straightforward to verify that $\phi = \{(\alpha_t, \beta_t), t \in [0, T]\}$ is a trading strategy with discounted value process C^* . By the continuity of C^* and of the stochastic integral process $\{\int_0^t \alpha_s dS_s^*, t \in [0, T]\}$, it follows on taking the limit as $t \rightarrow T$ that (4.73) also holds at $t = T$. Thus, we have P^* -a.s., for each $t \in [0, T]$,

$$V_t^*(\phi) = \alpha_t S_t^* + \beta_t = C_t^* = C_0 + \int_0^t \alpha_s dS_s^*,$$

which implies that ϕ is self-financing. Moreover, since $\{C_t^*, \mathcal{F}_t, t \in [0, T]\}$ is a martingale under P^* , ϕ is an admissible strategy. Finally, since $V_T(\phi) = C_T = (S_T - K)^+$, ϕ is a replicating strategy for the European call option with strike price K and expiration time T . The results of this section are summarized in the following theorem.

Theorem 4.5.1 *For each $t \in [0, T)$, let*

$$C_t^* = S_t^* \Phi \left(\frac{\log \left(\frac{S_t^*}{K^*} \right)}{\sigma \sqrt{T-t}} + \frac{1}{2} \sigma \sqrt{T-t} \right) - K^* \Phi \left(\frac{\log \left(\frac{S_t^*}{K^*} \right)}{\sigma \sqrt{T-t}} - \frac{1}{2} \sigma \sqrt{T-t} \right), \quad (4.76)$$

and let $C_T^ = (S_T^* - K^*)^+$. Define $\alpha_T = 1_{\{S_T^* > K^*\}}$, and let*

$$\alpha_t = \Phi \left(\frac{\log \left(\frac{S_t^*}{K^*} \right)}{\sigma \sqrt{T-t}} + \frac{1}{2} \sigma \sqrt{T-t} \right), \quad \text{for each } t \in [0, T), \quad (4.77)$$

$$\beta_t = C_t^* - \alpha_t S_t^* \quad \text{for each } t \in [0, T]. \quad (4.78)$$

Then $\phi = \{(\alpha_t, \beta_t), t \in [0, T]\}$ is a replicating strategy for the European call option with strike price K and expiration time T , and the arbitrage free price process for this option is given by $\{e^{rt} C_t^, t \in [0, T]\}$.*