

2

Binomial Model

In this chapter we consider a simple discrete financial market model called the binomial or Cox-Ross-Rubinstein (CRR) [8] model. In the context of this model, we derive the unique arbitrage free prices for a European and an American contingent claim based on this model.

2.1 Binomial or CRR Model

The CRR model is a simple discrete time model for a financial market. There are finitely many trading times $t = 0, 1, \dots, T$ ($T < \infty$), and two assets, a risky security called a stock, and a riskless security called a bond.

The *bond* is assumed to yield a constant rate of return $r \geq 0$ over each time period $(t-1, t]$ and so assuming the bond is valued at \$1 at time zero, the value of the bond at time t is given by

$$B_t = (1+r)^t, \quad t = 0, 1, \dots, T. \quad (2.1)$$

The *stock* price process is modeled as an exponential random walk such that S_0 is a strictly positive constant and

$$S_t = S_{t-1}\xi_t, \quad t = 1, 2, \dots, T, \quad (2.2)$$

where $\{\xi_t, t = 1, 2, \dots, T\}$ is a sequence of independent and identically distributed random variables with

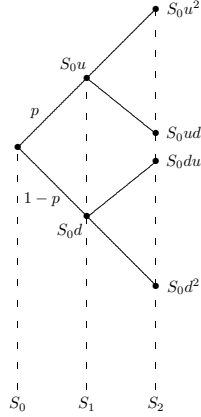
$$P(\xi_t = u) = p = 1 - P(\xi_t = d), \quad (2.3)$$

where $p \in (0, 1)$ and $0 < d < 1 + r < u$. The latter are assumed to avoid arbitrage opportunities in the primary market model and to ensure stock prices are random and strictly positive. Note that

$$S_t = S_0 \prod_{i=1}^t \xi_i, \quad t = 0, 1, \dots, T, \quad (2.4)$$

and one may represent the possible paths that S_t follows using a binary tree (see Figure 2.1). Note that there are only three distinct values for S_2 , i.e., the two middle dots in Figure 2.1 have the same value for S_2 . The points have been drawn as two distinct points to emphasize the fact that they may be reached by different paths, that is, through different values for the sequence S_0, S_1, S_2 . (For pricing and hedging of some contingent claims under the binomial model (e.g., a European contingent claim whose payoff depends only on the terminal value of the stock price or an American contingent claim whose payoff at time t depends only on the stock price at that time), one may use a so-called recombining tree in which only the distinct values of S_i are indicated (in particular, the two middle dots associated with values of S_2 in Figure 2.1 are identified). However, in general one needs the full path structure of the stock price process in order to price and hedge contingent claims, cf. Exercise 3 at the end of this Chapter).

For concreteness, and without loss of generality, we assume that the probability space $(\Omega, \mathcal{F}, P)$ on which our random variables are defined is such that Ω is the finite set of 2^T possible outcomes for the values of the stock price $(T+1)$ -tuple, $(S_0, S_1, S_2, \dots, S_T)$; $\mathcal{F}$ is the σ -algebra consisting of all possible subsets of Ω ; and P is the probability measure on $(\Omega, \mathcal{F})$ associated with the Bernoulli probability p . Then, for

Figure 2.1: Binary tree for $T = 2$

example,

$$P((S_0, S_1, \dots, S_T) = (S_0, S_0u, S_0u^2, \dots, S_0u^T)) = p^T. \quad (2.5)$$

The expectation operator under P will be denoted by $E[\cdot]$.

To describe the information available to the investor at time t , we introduce the σ -algebra generated by the stock prices up to and including time t , i.e, let

$$\mathcal{F}_t = \sigma\{S_0, S_1, \dots, S_t\}, \quad t = 0, 1, \dots, T. \quad (2.6)$$

In particular, with our concrete probability space, $\mathcal{F}_T = \mathcal{F}$.

A *trading strategy* (in the primary market) is a collection of pairs of random variables

$$\phi = \{(\alpha_t, \beta_t) : t = 1, 2, \dots, T\} \quad (2.7)$$

where the random variable α_t represents the number of shares of stock to be held over the time interval $(t-1, t]$ and the random variable β_t

represents the number of bonds to be held over the time interval $(t-1, t]$. For simplicity, we allow α_t, β_t to take any values in $\mathbb{R}$, in particular, we do not restrict to integer numbers of shares or bonds and we allow short selling of shares ($\alpha_t < 0$) and borrowing ($\beta_t < 0$). We think of trading occurring at time $t-1$ to determine the portfolio holdings (α_t, β_t) until the next trading time t . To avoid strategies that anticipate the future, it is assumed that α_t, β_t are $\mathcal{F}_{t-1}$ -measurable random variables for $t = 1, 2, \dots, T$. Thus, the holdings in stock and bond over the time period $(t-1, t]$ can only depend on the stock prices observed up to and including time $t-1$. We will restrict attention here to *self-financing* trading strategies, namely, those trading strategies ϕ such that

$$\alpha_t S_t + \beta_t B_t = \alpha_{t+1} S_t + \beta_{t+1} B_t, \quad t = 1, 2, \dots, T-1, \quad (2.8)$$

and the investor's initial wealth is equal to

$$W_0 = \alpha_1 S_0 + \beta_1 B_0. \quad (2.9)$$

We will simply refer to these as trading strategies, rather than using the longer term self-financing trading strategies. We say that a trading strategy ϕ represents a portfolio whose *value* at time t is given by $V_t(\phi)$, where

$$V_0(\phi) = \alpha_1 S_0 + \beta_1 B_0, \quad (2.10)$$

$$V_t(\phi) = \alpha_t S_t + \beta_t B_t, \quad t = 1, 2, \dots, T. \quad (2.11)$$

An *arbitrage opportunity* (in the primary market) is a trading strategy ϕ such that $V_0(\phi) = 0$, $V_T(\phi) \geq 0$ and $E[V_T(\phi)] > 0$. Note that, in the presence of the preceding conditions, the last condition is equivalent to $P(V_T(\phi) > 0) > 0$.

2.2 Pricing a European Contingent Claim

A *European contingent claim* is represented by a $\mathcal{F}_T$ -measurable random variable X . The value of this contingent claim at the expiration time T is X . For example, a European call option with strike price K and expiration date T is represented by $X = (S_T - K)^+ \equiv \max\{0, S_T - K\}$. The interpretation of X in this case is as follows. If $S_T > K$, the holder of the European call option will exercise it at T , and, after selling a share of stock just purchased for K , will make a profit of $S_T - K$ (ignoring whatever was paid for the option in the first place). Thus, in this case, the value of the option at T is $S_T - K$. On the other hand, if $S_T \leq K$, the holder of the option will not exercise it at T and its value is 0 at T . Similarly, a European put option with the same strike price and expiration date is represented by $X = (K - S_T)^+$.

A *replicating (or hedging) strategy* for a European contingent claim X is a trading strategy ϕ such that $V_T(\phi) = X$. If there exists such a replicating strategy, the contingent claim is said to be *attainable* (or *redundant*).

In this section, we derive the (initial) arbitrage free price for a European contingent claim. Existence of such a price depends on the existence of a so-called risk neutral probability and uniqueness depends on there being a replicating strategy for the contingent claim.

2.2.1 Single Period Case

We begin by examining the single period case where $T = 1$. For this case, we first show that there is a replicating strategy for any European contingent claim X . Given X , we seek a trading strategy $\phi = (\alpha_1, \beta_1)$, where α_1 and β_1 are constants, such that

$$V_1(\phi) \equiv \alpha_1 S_1 + \beta_1 B_1 = X. \quad (2.12)$$

Now S_1 has two possible values, $S_0 u, S_0 d$, and X is a function of S_1 , since it is $\mathcal{F}_1 = \sigma\{S_0, S_1\}$ -measurable. Let X^u denote the value of X when $S_1 = S_0 u$ and X^d denote the value of X when $S_1 = S_0 d$. Then considering these two possible outcomes, (2.12) yields two equations for the two deterministic unknowns α_1, β_1 :

$$\alpha_1 S_0 u + \beta_1 (1 + r) = X^u \quad (2.13)$$

$$\alpha_1 S_0 d + \beta_1 (1 + r) = X^d. \quad (2.14)$$

Solving for α_1, β_1 yields

$$\alpha_1 = \frac{X^u - X^d}{(u - d)S_0}, \quad (2.15)$$

$$\beta_1 = \frac{1}{1 + r} \left(\frac{uX^d - dX^u}{u - d} \right). \quad (2.16)$$

The right member of (2.15) is sometimes written informally as $\frac{\delta X}{\delta S}$. The initial wealth needed to finance this strategy (sometimes called the manufacturing cost of the contingent claim) is

$$\begin{aligned} V_0 &= \alpha_1 S_0 + \beta_1 B_0 \\ &= \frac{1}{(1 + r)(u - d)} ((1 + r - d)X^u + (u - (1 + r))X^d) \\ &= \frac{1}{1 + r} (p^* X^u + (1 - p^*) X^d) \\ &= E^{p^*}[X^*], \end{aligned} \quad (2.17)$$

where $p^* = \frac{1+r-d}{u-d}$, $E^{p^*}[\cdot]$ denotes the expectation operator with $p = p^*$, and $X^* = X/(1 + r)$. Note that $E^{p^*}[S_1] = (1 + r)S_0$ and so the discounted stock price process

$$\left\{ S_0, \frac{1}{1 + r} S_1 \right\}$$

is a martingale under $p = p^*$ (relative to the filtration $\{\mathcal{F}_t\}$). Thus, under $p = p^*$, the average rate of return of the risky asset is the same as that of the riskless asset. For this reason p^* is called the *risk neutral probability*. (A person is said to be “risk averse” if he/she prefers

the expected value of a payoff to the random payoff itself. A person is said to be “risk preferring” if he/she prefers the random payoff to the expected value of the payoff. A person is “risk neutral” if he/she is neither risk averse nor risk preferring, in other words he/she is indifferent, having no preference for the expected payoff versus the random payoff. The probability p^* is called a risk neutral probability because under the assumption that $p = p^*$, a risk neutral investor would be indifferent to the choice at time zero of investing S_0 in stock or bonds, since both investments have the same expected payoff at time 1 under $p = p^*$.

It is important to realize that computing expectations under $p = p^*$ is a mathematical device. We are not assuming that the stock price actually moves according to this probability. That is, p^* may be unrelated to the subjective probability p that we associate with the binomial model for movements in the stock price.

For the next theorem, we need the notion of an arbitrage opportunity in the market consisting of the stock, bond and contingent claim. For this, we suppose that the price of the contingent claim at time zero is C_0 . A *trading strategy* in stock, bond and the contingent claim is a triple $\psi = (\alpha_1, \beta_1, \gamma_1)$ of $\mathcal{F}_0$ -measurable random variables (these will actually be constants), where α_1 represents the number of shares of stock held over $(0, 1]$, β_1 represents the number of bonds held over $(0, 1]$ and γ_1 represents the number of units of the contingent claim held over $(0, 1]$. The initial value of the portfolio associated with ψ is $V_0(\psi) = \alpha_1 S_0 + \beta_1 B_0 + \gamma_1 C_0$. The value of this portfolio at time one is $V_1(\psi) = \alpha_1 S_1 + \beta_1 B_1 + \gamma_1 X$. An *arbitrage opportunity* in the stock-bond-contingent claim market is a trading strategy $\psi = (\alpha_1, \beta_1, \gamma_1)$ such that $V_0(\psi) = 0$, $V_1(\psi) \geq 0$ and $E[V_1(\psi)] > 0$.

Theorem 2.2.1 $V_0 = E^{p^*}[X^*]$ is the unique arbitrage free initial price for the European contingent claim X .

PROOF. Let $\phi^* = (\alpha_1^*, \beta_1^*)$ denote the replicating strategy (in stock and bond) for the contingent claim X , cf. (2.15)-(2.16).

First we show that if the initial price C_0 of the contingent claim is anything other than V_0 , then there is an arbitrage opportunity in the stock-bond-contingent claim market. Suppose $C_0 > V_0$. Then an investor starting with zero initial wealth could sell one contingent claim ($\gamma_1 = -1$), invest V_0 in the replicating strategy $\phi^* = (\alpha_1^*, \beta_1^*)$ and invest the remainder, $C_0 - V_0$, in bond. Thus, his trading strategy in stock, bond and contingent claim is $(\alpha_1^*, \beta_1^* + C_0 - V_0, -1)$. This has an initial value of zero and its value at time one is

$$\alpha_1^* S_1 + \beta_1^* B_1 + (C_0 - V_0) B_1 - X. \quad (2.18)$$

But the strategy (α_1^*, β_1^*) was chosen so that

$$\alpha_1^* S_1 + \beta_1^* B_1 = X, \quad (2.19)$$

and so it follows that the value at time one of the stock-bond-contingent claim portfolio is

$$(C_0 - V_0) B_1 > 0. \quad (2.20)$$

Thus, this represents an arbitrage opportunity. Similarly, if $C_0 < V_0$, then the investor can use the strategy $(-\alpha_1^*, -\beta_1^* + V_0 - C_0, 1)$ to create an arbitrage opportunity.

Now we show that if $C_0 = V_0$, then there is no arbitrage opportunity in the stock-bond-contingent claim market. Suppose that $\psi = (\alpha_1, \beta_1, \gamma_1)$ is a trading strategy in stock, bond and the contingent claim, with an initial value $V_0(\psi) = \alpha_1 S_0 + \beta_1 B_0 + \gamma_1 C_0$ of zero and non-negative value $V_1(\psi)$ at time one. The value of the portfolio at time one is

$$V_1(\psi) = \alpha_1 S_1 + \beta_1 B_1 + \gamma_1 X \quad (2.21)$$

and so

$$\begin{aligned}
E^{p^*}[V_1(\psi)] &= \alpha_1 E^{p^*}[S_1] + \beta_1(1+r) + \gamma_1 E^{p^*}[X] \\
&= \alpha_1(1+r)S_0 + \beta_1(1+r) + \gamma_1(1+r)C_0 \\
&= (1+r)V_0(\psi) \\
&= 0,
\end{aligned}$$

where we have used the martingale property of the discounted stock price process under p^* and the definition of $C_0 = V_0 = E^{p^*}[X]/(1+r)$. Now, since the probability measure P^* on $\mathcal{F}_1$, associated with using $p^* \in (0,1)$ in place of p , gives positive probability to both of the possible values of the non-negative random variable $V_1(\psi)$, it follows that $P^*(V_1(\psi) = 0) = 1$. Then, since P^* has the same sets of probability zero (or one) as P , $P(V_1(\psi) = 0) = 1$. Thus, there cannot be an arbitrage opportunity. $\square$

Examination of the proof of Theorem 2.2.1 reveals that existence of an arbitrage free price for the contingent claim depends on the martingale property of $\{S_0, \frac{1}{1+r}S_1\}$ under $p = p^*$ and uniqueness of the price depends on the existence of a replicating strategy for the contingent claim. These aspects will be true more generally.

2.2.2 Multi-Period Case

We now consider the general binomial model where T is any fixed positive integer. In this subsection p^* has the same value as in the single period case, namely, $p^* = \frac{1+r-d}{u-d}$.

We first show that there is a replicating strategy for a European contingent claim X . For this, given X , we seek a trading strategy $\phi = \{(\alpha_t, \beta_t), t = 1, \dots, T\}$ such that

$$V_T(\phi) \equiv \alpha_T S_T + \beta_T B_T = X. \quad (2.22)$$

This is developed by working backwards through the binary tree.

Let $V_T = X$. Since X is an $\mathcal{F}_T$ -measurable random variable and $(S_0, S_1, \dots, S_T)$ can take only finitely many possible values,

$$V_T = f(S_0, S_1, \dots, S_T)$$

for some real-valued function f of $T+1$ variables. Firstly, suppose we condition on knowing $S_0, S_1, \dots, S_{T-1}$. Then the cost and associated trading strategy for manufacturing the contingent claim over the time period $(T-1, T]$ can be computed in a very similar manner to that for the single period model. Given $S_0, S_1, \dots, S_{T-1}$, there are two possible values for V_T at time T , depending on whether $S_T = S_{T-1}u$ or $S_T = S_{T-1}d$. Denote these two values by V_T^u and V_T^d . In fact, $V_T^u = f(S_0, S_1, \dots, S_{T-1}, S_{T-1}u)$ and $V_T^d = f(S_0, S_1, \dots, S_{T-1}, S_{T-1}d)$. Note that these are $\mathcal{F}_{T-1}$ -measurable random variables (the notation V_T^u, V_T^d hides this fact, but has the advantage that it makes the formulas for the replicating strategy appear simpler). If the contingent claim is a European call option with strike price K and expiration date T , then $X = (S_T - K)^+$ and $V_T^u = (S_{T-1}u - K)^+$, $V_T^d = (S_{T-1}d - K)^+$.

Now, for any European contingent claim X , by similar analysis to that for the single period case, to ensure that $V_T(\phi) = X$, we obtain the following allocations for the time period $(T-1, T]$:

$$\alpha_T = \frac{V_T^u - V_T^d}{(u-d)S_{T-1}} \quad (2.23)$$

$$\beta_T = \frac{1}{(1+r)^T} \left(\frac{uV_T^d - dV_T^u}{u-d} \right) \quad (2.24)$$

and the capital required at time $T-1$ to finance these allocations in a

self-financing manner is

$$\begin{aligned}
 V_{T-1} &= \frac{1}{1+r} (p^* V_T^u + (1-p^*) V_T^d) \\
 &= \frac{1}{1+r} E^{p^*} [V_T | \mathcal{F}_{T-1}], \\
 &= \frac{1}{1+r} E^{p^*} [X | \mathcal{F}_{T-1}],
 \end{aligned} \tag{2.25}$$

where $p^* = \frac{1+r-d}{u-d}$ and $E^{p^*}[\cdot | \mathcal{F}_{T-1}]$ denotes the conditional expectation, given $\mathcal{F}_{T-1}$, when the subjective probability p is replaced by p^* .

We can find a trading strategy $\phi = \{(\alpha_t, \beta_t), t = 1, 2, \dots, T\}$ with associated value process $\{V_t(\phi), t = 0, 1, \dots, T\}$ such that $V_T = X$ by proceeding inductively backwards through the binary tree as follows. For the induction step, fix $t \in \{1, \dots, T-1\}$ and assume that (self-financing) allocations $(\alpha_{t+1}, \beta_{t+1}), \dots, (\alpha_T, \beta_T)$ have been determined for the time periods $(t, t+1], \dots, (T-1, T]$ with values $V_t, \dots, V_{T-1}$, at times $t, t+1, \dots, T-1$, respectively, such that $V_s = \frac{1}{(1+r)^{T-s}} E^{p^*} [X | \mathcal{F}_s]$ for $s = t, t+1, \dots, T-1$, and the value of (α_T, β_T) at time T is $V_T = X$. Given $S_0, S_1, \dots, S_{t-1}$, the holdings α_t and β_t for the time period $(t-1, t]$ are chosen so that the value associated with these holdings at time t is the same as that of the random variable V_t , i.e., letting V_t^u and V_t^d denote the two possible values of V_t given $S_0, S_1, \dots, S_{t-1}$, define

$$\alpha_t = \frac{V_t^u - V_t^d}{(u-d)S_{t-1}}, \tag{2.26}$$

$$\beta_t = \frac{1}{(1+r)^t} \left(\frac{uV_t^d - dV_t^u}{u-d} \right). \tag{2.27}$$

One can readily check, using the induction hypothesis, that the capital needed at time $t-1$ to finance these holdings in a self-financing manner

is

$$\begin{aligned}
 V_{t-1} &= \frac{1}{1+r} E^{p^*} [V_t | \mathcal{F}_{t-1}] \\
 &= \frac{1}{(1+r)^{T-t+1}} E^{p^*} [E^{p^*} [X | \mathcal{F}_t] | \mathcal{F}_{t-1}] \\
 &= \frac{1}{(1+r)^{T-t+1}} E^{p^*} [X | \mathcal{F}_{t-1}].
 \end{aligned}$$

This completes the induction step and so it follows that the above procedure constructs a (self-financing) trading strategy, $\phi = \{(\alpha_t, \beta_t) : t = 1, \dots, T\}$ with value process $\{V_t(\phi), t = 0, 1, \dots, T\}$ satisfying

$$V_t(\phi) = \frac{1}{(1+r)^{T-t}} E^{p^*} [X | \mathcal{F}_t] \tag{2.28}$$

for $t = 0, 1, \dots, T$. In particular, $V_T(\phi) = X$ and

$$V_0 = \frac{1}{(1+r)^T} E^{p^*} [X]. \tag{2.29}$$

Here we have used the fact that $\mathcal{F}_0$ is trivial, being generated by the constant value S_0 .

To prove that there exists an arbitrage free price for the contingent claim X , we need the following properties. For this, let $S_t^* = S_t/B_t = S_t/(1+r)^t$, $t = 0, 1, \dots, T$. The process $S^* = \{S_t^*, t = 0, 1, \dots, T\}$ is called the *discounted stock price process*.

Lemma 2.2.2 *The process $\{S_t^*, \mathcal{F}_t, t = 0, 1, \dots, T\}$ is a martingale when probabilities are computed using p^* in place of p .*

PROOF. Clearly S_t^* is $\mathcal{F}_t$ -measurable, and S_t^* has finite mean for each t , since S_t^* takes only finitely many values. To verify the conditional expectation property, fix $t \in \{1, \dots, T\}$. Then, using the fact that $\mathcal{F}_{t-1}$ is generated by $S_0, S_1, \dots, S_{t-1}$ and ξ_t is independent of this σ -algebra,

we have

$$\begin{aligned}
E^{p^*}[S_t^* | \mathcal{F}_{t-1}] &= \frac{1}{(1+r)^t} E^{p^*}[S_{t-1}\xi_t | \mathcal{F}_{t-1}] \\
&= \frac{1}{(1+r)^t} S_{t-1} E^{p^*}[\xi_t] \\
&= \frac{S_{t-1}^*}{1+r} (p^*u + (1-p^*)d) \\
&= S_{t-1}^*,
\end{aligned} \tag{2.30}$$

where we have used the definition of $p^* = \frac{1+r-d}{u-d}$ to obtain the last line. $\square$

Lemma 2.2.3 *Let ϕ be a trading strategy in stock and bond with value process $\{V_t(\phi), t = 1, \dots, T\}$ and discounted value process $\{V_t^*(\phi) = V_t(\phi)/(1+r)^t, t = 0, 1, \dots, T\}$. Then $\{V_t^*(\phi), \mathcal{F}_t, t = 0, 1, \dots, T\}$ is a martingale when probabilities are computed using p^* in place of p .*

PROOF. Let $\phi = \{(\alpha_t, \beta_t), t = 1, \dots, T\}$. Note that $V_t^*(\phi) = \alpha_t S_t^* + \beta_t$ and $\alpha_t, \beta_t \in \mathcal{F}_{t-1}$ for $t = 1, \dots, T$, and $V_0^*(\phi) = \alpha_1 S_0^* + \beta_1$. It is straightforward to show that $V_t^*(\phi)$ is $\mathcal{F}_t$ -measurable and integrable for $t = 0, 1, \dots, T$. Recall from Lemma 2.2.2 that $\{S_t^*, \mathcal{F}_t, t = 0, 1, \dots, T\}$ is a martingale when p^* is used in place of p . Therefore, for $t = 1, \dots, T$,

$$\begin{aligned}
E^{p^*}[V_t^*(\phi) | \mathcal{F}_{t-1}] &= \alpha_t E^{p^*}[S_t^* | \mathcal{F}_{t-1}] + \beta_t \\
&= \alpha_t S_{t-1}^* + \beta_t.
\end{aligned} \tag{2.31}$$

By factoring out $1/(1+r)^{t-1}$ and using the self-financing property of ϕ , it follows that for $t = 2, \dots, T$

$$\begin{aligned}
E^{p^*}[V_t^*(\phi) | \mathcal{F}_{t-1}] &= \frac{1}{(1+r)^{t-1}} (\alpha_t S_{t-1} + \beta_t B_{t-1}) \\
&= \frac{1}{(1+r)^{t-1}} (\alpha_{t-1} S_{t-1} + \beta_{t-1} B_{t-1}) \\
&= V_{t-1}^*(\phi).
\end{aligned}$$

For $t = 1$,

$$\begin{aligned}
E^{p^*}[V_1^*(\phi) | \mathcal{F}_0] &= \alpha_1 S_0 + \beta_1 B_0 \\
&= V_0(\phi) = V_0^*(\phi),
\end{aligned}$$

by definition. Hence the desired martingale property holds. $\square$

The following theorem is the multi-period analogue of the single period Theorem 2.2.1. First we specify the notion of arbitrage in stock, bond and contingent claim to be used in the multi-period context. Let C_0 be the price charged for the contingent claim at time zero. Since we are only specifying a price for the contingent claim at time zero, trading in the contingent claim will only be allowed initially, whereas changes in the stock and bond holdings can occur at each of the times $t = 0, 1, \dots, T-1$. A *trading strategy* in stock, bond and the contingent claim, is a collection $\psi = \{(\alpha_t, \beta_t), t = 1, 2, \dots, T; \gamma_1\}$ where for $t = 1, 2, \dots, T$, α_t, β_t are $\mathcal{F}_{t-1}$ -measurable random variables representing the holdings in stock and bond, respectively, to be held over the time interval $(t-1, t]$, and γ_1 is an $\mathcal{F}_0$ -measurable random variable (actually a constant) representing the number of units of the contingent claim to be held over the time interval $(0, T]$. The trading strategy must be self-financing, i.e., its initial value is

$$V_0(\psi) = \alpha_1 S_0 + \beta_1 B_0 + \gamma_1 C_0, \tag{2.32}$$

and at each time $t = 1, \dots, T-1$,

$$\alpha_t S_t + \beta_t B_t = \alpha_{t+1} S_t + \beta_{t+1} B_t. \tag{2.33}$$

The last equation does not involve the contingent claim since this is not traded after time zero. The value of the portfolio at time T is

$$V_T(\psi) = \alpha_T S_T + \beta_T B_T + \gamma_1 X. \tag{2.34}$$

An *arbitrage opportunity* in the stock-bond-contingent claim market is a trading strategy ψ such that $V_0(\psi) = 0$, $V_T(\psi) \geq 0$ and $E[V_T(\psi)] > 0$.

Theorem 2.2.4 *Let $X^* = X/(1+r)^T$. Then $V_0 = E^{p^*}[X^*]$ is the unique arbitrage free initial price for the European contingent claim X .*

PROOF. The proof is very similar to that of Theorem 2.2.1.

Let $\phi^* = \{(\alpha_t^*, \beta_t^*) : t = 1, \dots, T\}$ denote the replicating strategy for the contingent claim X obtained by applying the inductive procedure described above (cf. (2.26)–(2.27)). Then the value process $V = \{V_t(\phi^*), t = 0, 1, \dots, T\}$ for ϕ^* satisfies (2.28) for $t = 0, 1, 2, \dots, T$. Note that, by (2.29), V_0 is the initial value of this replicating strategy

We use the existence of the replicating strategy to show that if the initial price C_0 of the contingent claim is not V_0 then there is an arbitrage opportunity in the stock-bond-contingent claim market. Suppose $C_0 > V_0$. Then an investor could sell one contingent claim initially, use V_0 of the proceeds to invest in the stock-bond replicating strategy ϕ^* , and buy $C_0 - V_0$ additional bonds at time zero and hold them over the entire period $(0, T]$. Thus, the trading strategy is $\psi = \{(\alpha_t^*, \beta_t^* + C_0 - V_0), t = 1, 2, \dots, T; \gamma_1 = -1\}$. This has initial value $V_0(\psi) = \alpha_1^* S_0 + \beta_1^* + C_0 - V_0 - C_0 = 0$, since $B_0 = 1$ and $V_0 = V_0(\phi^*) = \alpha_1^* S_0 + \beta_1^*$. The value of this portfolio at time T is

$$\begin{aligned} V_T(\psi) &= \alpha_T^* S_T + \beta_T^* B_T + (C_0 - V_0) B_T - X \\ &= X + (C_0 - V_0) B_T - X = (C_0 - V_0) B_T > 0, \end{aligned} \quad (2.35)$$

where we have used the fact that $\{(\alpha_t^*, \beta_t^*), t = 1, 2, \dots, T\}$ is a replicating strategy for the contingent claim X and so has value X at time T . Thus, ψ is an arbitrage opportunity. Similarly, if $C_0 < V_0$, the strategy $-\psi$ is an arbitrage opportunity.

Now suppose $C_0 = V_0$. We show that there is no arbitrage opportunity in stock, bond and contingent claim trading. Let $\psi = \{(\alpha_t, \beta_t), t = 1, 2, \dots, T; \gamma_1\}$ be a trading strategy in stock, bond, and contingent claim with an initial value of zero and final value $V_T(\psi)$ that is a non-

negative random variable. Then,

$$0 = V_0(\psi) = \alpha_1 S_0 + \beta_1 B_0 + \gamma_1 C_0, \quad (2.36)$$

$$0 \leq V_T(\psi) = \alpha_T S_T + \beta_T B_T + \gamma_1 X. \quad (2.37)$$

Note that $\phi = \{(\alpha_t, \beta_t) : t = 1, 2, \dots, T\}$ is a trading strategy in stock and bond. Using the martingale property of $\{V_t^*(\phi), \mathcal{F}_t, t = 0, 1, \dots, T\}$ with p^* in place of p , on setting $V_t^*(\psi) = V_t(\psi)/(1+r)^t$ for $t = 0, 1, \dots, T$, we have

$$\begin{aligned} \frac{1}{(1+r)^T} E^{p^*}[V_T(\psi)] &= E^{p^*}[V_T^*(\psi)] \\ &= E^{p^*}[V_T^*(\phi)] + E^{p^*}[\gamma_1 X^*] \\ &= E^{p^*}[V_0^*(\phi)] + \gamma_1 E^{p^*}[X^*] \\ &= \alpha_1 S_0^* + \beta_1 B_0^* + \gamma_1 C_0 \\ &= V_0(\psi) = 0 \end{aligned}$$

where we have used the facts that $\gamma_1 \in \mathcal{F}_0$, $C_0 = E^{p^*}[X^*]$, $S_0^* = S_0$, $B_0^* = B_0$, and $V_0(\psi) = 0$. Since it was assumed that $V_T(\psi) \geq 0$ and the probability measure associated with p^* gives positive probability to all possible values of $V_T(\psi)$, it follows that $V_T(\psi) = 0$ and hence $E[V_T(\psi)] = 0$. Thus, there cannot be any arbitrage opportunity when $C_0 = V_0$. $\square$

2.3 Pricing an American Contingent Claim

An *American contingent claim* (ACC) is represented by a (finite) sequence $Y = \{Y_t, t = 0, 1, \dots, T\}$ of real-valued random variables such that $Y_t \in \mathcal{F}_t$ for $t = 0, 1, 2, \dots, T$. The random variable Y_t , $t = 0, 1, 2, \dots, T$, is interpreted as the payoff for the claim if the owner cashes it in at time t . The time at which the owner cashes in the claim is required to be a stopping time taking values in $\{0, 1, \dots, T\}$, i.e.,

a random variable $\tau : \Omega \longrightarrow \{0, 1, \dots, T\}$ such that $\{\tau = t\} \in \mathcal{F}_t$, $t = 0, 1, 2, \dots, T$. For $s, t \in \{0, 1, \dots, T\}$ such that $s \leq t$, let $\mathcal{T}_{[s,t]}$ denote the set of integer-valued stopping times that take values in the interval $[s, t]$. An example of an American contingent claim is an *American call option* with strike price K which has payoff $Y_t = (S_t - K)^+$ at time t , $t = 0, 1, 2, \dots, T$. Note that if $S_t \leq K$, cashing in the contingent claim at time t has an equivalent payoff to that obtained by not exercising the option at all. We have adopted this convention so that we can use one framework for treating all contingent claims, including options and contracts.

An important feature of an American contingent claim is that the buyer and the seller of such a derivative have different actions available to them — the buyer may cash in the claim at any stopping time $\tau \in \mathcal{T}_{[0,T]}$, whereas the seller seeks protection from the risk associated with all possible choices of the stopping time τ by the buyer. As with the pricing of European contingent claims, for the pricing of American contingent claims, an essential role will be played by a trading strategy that hedges the risk of the seller of an American contingent claim. However, unlike the European contingent claim setting, the seller will not always be able to exactly replicate the payoff of the American contingent claim at all times t . Instead, the seller of an American contingent claim seeks a *super hedging strategy* (also called a *perfect hedging strategy*) which is a (self-financing) trading strategy ϕ whose value is at least as great as the payoff of the American contingent claim at each time t .

More precisely, let $Y = \{Y_0, Y_1, \dots, Y_T\}$ be the payoff sequence for an American contingent claim (abbreviated as ACC). For $t = 0, 1, \dots, T$, let U_t denote the minimum amount of wealth that the seller of the ACC must have at time t in order to cover the payoff if the buyer cashes in the claim at some stopping time $\tau \in \mathcal{T}_{[t,T]}$. A super hedging strategy for the

seller is a (self-financing) trading strategy $\phi = \{(\alpha_t, \beta_t) : t = 1, 2, \dots, T\}$ with value $V_t(\phi)$ at time $t = 0, 1, \dots, T$, such that $U_t \leq V_t(\phi)$ for $t = 0, 1, 2, \dots, T$. Such a ϕ can be constructed stepwise by proceeding backwards through the binary tree. In order to see this, note that $U_T = Y_T$. Conditioned on $\mathcal{F}_{T-1}$ (i.e., given $S_0, S_1, \dots, S_{T-1}$), let U_T^u denote the amount that the seller of the ACC must cover at time T if the value of the stock at that time is $S_{T-1}u$ and let U_T^d denote the amount that the seller must cover at time T if the value of the stock at time T is $S_{T-1}d$. Thus, in the same manner as for the case of the European contingent claim, conditioned on $\mathcal{F}_{T-1}$, the minimum amount of wealth needed at time $T-1$ to cover the possible payoff of the ACC at time T is

$$\frac{1}{1+r} E^{p^*} [U_T \mid \mathcal{F}_{T-1}],$$

and there exist $\mathcal{F}_{T-1}$ -measurable allocations to stock and bond for the time interval $(T-1, T]$, which we denote by $(\tilde{\alpha}_T, \tilde{\beta}_T)$, such that

$$\tilde{\alpha}_T S_T + \tilde{\beta}_T B_T = U_T, \quad (2.38)$$

$$\tilde{\alpha}_T S_{T-1} + \tilde{\beta}_T B_{T-1} = \frac{1}{1+r} E^{p^*} [U_T \mid \mathcal{F}_{T-1}]. \quad (2.39)$$

Now,

$$U_{T-1} = \max \left\{ Y_{T-1}, \frac{1}{1+r} E^{p^*} [U_T \mid \mathcal{F}_{T-1}] \right\},$$

so that it is possible to cover the payoff Y_{T-1} associated with the buyer cashing in the claim at time $T-1$ and to have sufficient wealth to produce a value at time T that is at least as large as the time T -payoff $U_T = Y_T$ of the ACC. Let

$$\tilde{\delta}_T = U_{T-1} - \frac{1}{1+r} E^{p^*} [U_T \mid \mathcal{F}_{T-1}]. \quad (2.40)$$

Then $\tilde{\delta}_T$ is the excess wealth in U_{T-1} over what is needed to cover the claim payoff at time T . Note that there is no excess if $Y_{T-1} < U_{T-1}$. Proceeding backwards inductively through the tree and repeating a very similar argument at each stage to that applied for the time interval

$(T-1, T]$, we see that for $t = 0, 1, \dots, T-1$, conditioned on $\mathcal{F}_t$, the amount of wealth needed at time t to cover possible payoff of the claim in $[t, T]$ is

$$U_t = \max \left\{ Y_t, \frac{1}{1+r} E^{p^*} [U_{t+1} \mid \mathcal{F}_t] \right\}.$$

Moreover, conditioned on $\mathcal{F}_t$ (i.e., given $S_0, S_1, \dots, S_t$), and letting U_{t+1}^u and U_{t+1}^d denote the two possible values of U_{t+1} corresponding to whether $S_{t+1} = S_t u$ or $S_{t+1} = S_t d$, the allocations in stock and bond, $(\tilde{\alpha}_{t+1}, \tilde{\beta}_{t+1})$ over $(t, t+1]$, that have value

$$\frac{1}{1+r} E^{p^*} [U_{t+1} \mid \mathcal{F}_t],$$

at time t and value U_{t+1} at time $t+1$ are given by

$$\tilde{\alpha}_{t+1} = \frac{U_{t+1}^u - U_{t+1}^d}{(u-d)S_t} \quad (2.41)$$

$$\tilde{\beta}_{t+1} = \frac{1}{(1+r)^{t+1}} \left(\frac{uU_{t+1}^d - dU_{t+1}^u}{u-d} \right). \quad (2.42)$$

Let

$$\tilde{\delta}_{t+1} = U_t - \frac{1}{1+r} E^{p^*} [U_{t+1} \mid \mathcal{F}_t]. \quad (2.43)$$

Then $\tilde{\delta}_{t+1}$ is the excess wealth in U_t over what is needed to cover possible payoff of the claim in $[t+1, T]$. Note from the definition of U_t that $\tilde{\delta}_{t+1} = 0$ if $Y_t < U_t$.

Assuming that $U_t, t = 0, 1, \dots, T$ and $\tilde{\alpha}_t, \tilde{\beta}_t, \tilde{\delta}_t, t = 1, 2, \dots, T$ are defined as above, let

$$\alpha_t^* = \tilde{\alpha}_t, \quad (2.44)$$

$$\beta_t^* = \tilde{\beta}_t + \sum_{s=1}^t \frac{\tilde{\delta}_s}{B_{s-1}}, \quad (2.45)$$

for $t = 1, 2, \dots, T$. For this, note that $\tilde{\delta}_s$ is the excess available at time $s-1$, over the amount $\frac{1}{1+r} E^{p^*} [U_s \mid \mathcal{F}_{s-1}]$ that is needed at time $s-1$ to fund $(\tilde{\alpha}_s, \tilde{\beta}_s)$. We think of this excess as being invested in bonds for

the time period $(s-1, T]$. The cost of each bond is B_{s-1} at time $s-1$ and so the number of bonds bought with $\tilde{\delta}_s$ will be $\tilde{\delta}_s/B_{s-1}$.

By construction, for $t = 1, \dots, T$,

$$\tilde{\alpha}_t S_t + \tilde{\beta}_t B_t = U_t.$$

The strategy $\phi^* = \{(\alpha_t^*, \beta_t^*), t = 1, \dots, T\}$ for trading in stock and bond is self-financing since for $t = 1, \dots, T-1$,

$$\begin{aligned} \alpha_{t+1}^* S_t + \beta_{t+1}^* B_t &= \tilde{\alpha}_{t+1} S_t + \tilde{\beta}_{t+1} B_t + \left(\sum_{s=1}^{t+1} \frac{\tilde{\delta}_s}{B_{s-1}} \right) B_t \\ &= \frac{1}{1+r} E^{p^*} [U_{t+1} \mid \mathcal{F}_t] + \left(\sum_{s=1}^t \frac{\tilde{\delta}_s}{B_{s-1}} \right) B_t + \tilde{\delta}_{t+1} \\ &= U_t + \left(\sum_{s=1}^t \frac{\tilde{\delta}_s}{B_{s-1}} \right) B_t \\ &= \tilde{\alpha}_t S_t + \tilde{\beta}_t B_t + \left(\sum_{s=1}^t \frac{\tilde{\delta}_s}{B_{s-1}} \right) B_t \\ &= \alpha_t^* S_t + \beta_t^* B_t. \end{aligned}$$

The initial value of this trading strategy is

$$V_0(\phi^*) = \alpha_1^* S_0 + \beta_1^* B_0 = U_0$$

and its value at time $t = 1, 2, \dots, T$ is

$$V_t(\phi^*) = \tilde{\alpha}_t S_t + \tilde{\beta}_t B_t + \left(\sum_{s=1}^t \frac{\tilde{\delta}_s}{B_{s-1}} \right) B_t \geq U_t.$$

Let

$$\tau^* = \min\{s \geq 0 : U_s = Y_s\}. \quad (2.46)$$

Note that since $Y_s \leq U_s$ for $s = 0, 1, \dots, T$, $Y_s < U_s$ for $0 \leq s < \tau^*$, and so $\tilde{\delta}_s = 0$ for $1 \leq s \leq \tau^*$, and hence

$$V_t(\phi^*) = U_t \quad \text{for } 0 \leq t \leq \tau^*.$$

For $t = 0, 1, \dots, T$, define the discounted random variables

$$Y_t^* = (1+r)^{-t}Y_t, \quad U_t^* = (1+r)^{-t}U_t. \quad (2.47)$$

Then for $t = 0, 1, \dots, T-1$,

$$U_t^* = \max \{Y_t^*, E^{p^*}[U_{t+1}^* | \mathcal{F}_t]\}, \quad (2.48)$$

and $U_T^* = Y_T^*$.

Remark. As given by the formulas above, $\{U_t^*, t = 0, 1, \dots, T\}$ is called the *Snell envelope* of $\{Y_t^*, t = 0, 1, \dots, T\}$.

Lemma 2.3.1

- (i) When p^* is used in place of p , $U^* = \{U_t^*, \mathcal{F}_t, t = 0, 1, \dots, T\}$ is the smallest supermartingale such that $U_t^* \geq Y_t^*$ for $t = 0, 1, \dots, T$,
- (ii) $U_t^* = \max_{\tau \in \mathcal{T}_{[t, T]}} E^{p^*}[Y_\tau^* | \mathcal{F}_t]$, $t = 0, 1, \dots, T$,
- (iii) $\tau^*(t) \equiv \min\{v \geq t : U_v^* = Y_v^*\}$ achieves the maximum in the right side of the equality in (ii) for $t = 0, 1, \dots, T$.

PROOF. Throughout this proof, all expectations and conditional expectations are to be computed with p^* in place of p . For (i), the supermartingale property and the inequality are immediate consequences of (2.48). To show that U^* is the smallest supermartingale satisfying the inequality, suppose that

$$W = \{W_t, \mathcal{F}_t, t = 0, 1, \dots, T\}$$

is another supermartingale such that $W_t \geq Y_t^*$, $t = 0, 1, \dots, T$. Then $W_T \geq Y_T^* = U_T^*$. For a proof by backwards induction, suppose that for some $t \in \{0, 1, \dots, T-1\}$, $W_{t+1} \geq U_{t+1}^*$. Then by the supermartingale property of W ,

$$W_t \geq E^{p^*}[W_{t+1} | \mathcal{F}_t] \geq E^{p^*}[U_{t+1}^* | \mathcal{F}_t].$$

Since, by assumption $W_t \geq Y_t^*$, W_t is greater than or equal to the maximum of Y_t^* and $E^{p^*}[U_{t+1}^* | \mathcal{F}_t]$, which equals U_t^* . This completes the induction step and so $W_t \geq U_t^*$ for $t = T, T-1, \dots, 1, 0$.

We prove (ii) and (iii) together, using backwards induction again. For $t = T$, both (ii) and (iii) are easy to show since $\mathcal{T}_{[T, T]} = \{T\}$, Y_T^* is $\mathcal{F}_T$ -measurable, and $U_T^* = Y_T^*$. For the induction step, assume that for some $s \in \{0, 1, \dots, T-1\}$ both (ii) and (iii) hold for $t = s+1, s+2, \dots, T$. By (2.48) with $t = s$, and (ii) for $t = s+1$, for each $\tau \in \mathcal{T}_{[s+1, T]}$ we have

$$\begin{aligned} U_s^* &\geq \max \{Y_s^*, E^{p^*}[E^{p^*}[Y_\tau^* | \mathcal{F}_{s+1}] | \mathcal{F}_s]\} \\ &= \max \{Y_s^*, E^{p^*}[Y_\tau^* | \mathcal{F}_s]\}. \end{aligned} \quad (2.49)$$

For $\sigma \in \mathcal{T}_{[s, T]}$,

$$Y_\sigma^* = 1_{\{\sigma=s\}}Y_s^* + 1_{\{\sigma \geq s+1\}}Y_{\sigma \vee (s+1)}^*.$$

Since $1_{\{\sigma \geq s+1\}} \in \mathcal{F}_s$,

$$\begin{aligned} E^{p^*}[Y_\sigma^* | \mathcal{F}_s] &= 1_{\{\sigma=s\}}Y_s^* + 1_{\{\sigma \geq s+1\}}E^{p^*}[Y_{\sigma \vee (s+1)}^* | \mathcal{F}_s] \\ &\leq \max \{Y_s^*, E^{p^*}[Y_{\sigma \vee (s+1)}^* | \mathcal{F}_s]\}. \end{aligned} \quad (2.50)$$

Since $\sigma \vee (s+1) \in \mathcal{T}_{[s+1, T]}$, it follows from (2.49) and (2.50) that

$$E^{p^*}[Y_\sigma^* | \mathcal{F}_s] \leq U_s^*,$$

and since $\sigma \in \mathcal{T}_{[s, T]}$ was arbitrary, we conclude that

$$U_s^* \geq \max_{\tau \in \mathcal{T}_{[s, T]}} E^{p^*}[Y_\tau^* | \mathcal{F}_s]. \quad (2.51)$$

The proof of (ii) and (iii) for $t = s$ will be complete once we verify that the maximum in the right member of (2.51) is achieved at $\tau = \tau^*(s)$ and the value of this maximum is U_s^* . By (2.48) with $t = s$ and (iii) for $t = s+1$,

$$\begin{aligned} U_s^* &= \max \left\{ Y_s^*, E^{p^*} \left[E^{p^*} [Y_{\tau^*(s+1)}^* | \mathcal{F}_{s+1}] | \mathcal{F}_s \right] \right\} \\ &= \max \left\{ Y_s^*, E^{p^*} [Y_{\tau^*(s+1)}^* | \mathcal{F}_s] \right\}. \end{aligned} \quad (2.52)$$

On $\{\tau^*(s) = s\}$, $Y_s^* = U_s^*$, and on $\{\tau^*(s) \geq s+1\}$, $Y_s^* < U_s^*$. Thus,

$$U_s^* = 1_{\{\tau^*(s)=s\}}Y_s^* + 1_{\{\tau^*(s)\geq s+1\}}E^{p^*}\left[Y_{\tau^*(s+1)}^* \mid \mathcal{F}_s\right].$$

Moreover, on $\{\tau^*(s) \geq s+1\}$, $\tau^*(s) = \tau^*(s+1)$ and so the above yields

$$U_s^* = E^{p^*}\left[Y_{\tau^*(s)}^* \mid \mathcal{F}_s\right],$$

which completes the proof of (ii) and (iii) for $t = s$. $\square$

Recall the definition of τ^* from (2.46) and note that

$$\tau^* = \tau^*(0).$$

Then we have the following.

Lemma 2.3.2 *When p^* is used in place of p , $\{U_{t \wedge \tau^*}^*, \mathcal{F}_t, t = 0, 1, \dots, T\}$ is a martingale.*

PROOF. Since τ^* is a stopping time, $U_{t \wedge \tau^*}^* \in \mathcal{F}_t$ for $t = 0, 1, \dots, T$ and since the probability space is finite, the integrability of $U_{t \wedge \tau^*}^*$ is automatic for each t . Fix $t \in \{0, 1, \dots, T-1\}$. On $\{\tau^* > t\}$, $Y_t^* < U_t^*$ and thus $U_t^* = E^{p^*}[U_{t+1}^* \mid \mathcal{F}_t]$ there. Thus,

$$\begin{aligned} U_{(t+1) \wedge \tau^*}^* - U_{t \wedge \tau^*}^* &= 1_{\{\tau^* > t\}}(U_{t+1}^* - U_t^*) \\ &= 1_{\{\tau^* > t\}}(U_{t+1}^* - E^{p^*}[U_{t+1}^* \mid \mathcal{F}_t]), \end{aligned} \quad (2.53)$$

and then taking conditional expectations with respect to $\mathcal{F}_t$ with p^* in place of p , using the fact that $\{\tau^* > t\} \in \mathcal{F}_t$, yields

$$E^{p^*}[U_{(t+1) \wedge \tau^*}^* - U_{t \wedge \tau^*}^* \mid \mathcal{F}_t] = 0.$$

$\square$

We now argue that U_0 is the unique arbitrage free initial price for the American contingent claim. For this, we need the notion of an arbitrage in a market where the stock and bond can be traded, and

the American contingent claim (ACC) can be bought or sold at time zero. For such a market, let the initial price of the ACC be a constant C_0 . There are two types of arbitrage opportunities, one for a seller and another for a buyer of the ACC. The *seller* of the ACC has an *arbitrage opportunity* if there is a trading strategy ϕ^s in stock and bond such that $V_0(\phi^s) = C_0$ and for all stopping times $\tau \in \mathcal{T}_{[0,T]}$,

$$V_\tau(\phi^s) - Y_\tau \geq 0 \quad \text{and} \quad E[V_\tau(\phi^s) - Y_\tau] > 0. \quad (2.54)$$

The *buyer* of the ACC has an *arbitrage opportunity* if there is a trading strategy ϕ^b such that $V_0(\phi^b) = -C_0$ and there exists a stopping time $\tau \in \mathcal{T}_{[0,T]}$ such that

$$V_\tau(\phi^b) + Y_\tau \geq 0 \quad \text{and} \quad E[V_\tau(\phi^b) + Y_\tau] > 0. \quad (2.55)$$

The price C_0 is arbitrage free if there is no arbitrage opportunity for a seller or buyer of the contingent claim at this price.

To take advantage of a seller's arbitrage opportunity, an investor could sell one ACC at time zero for C_0 and invest the proceeds C_0 according to the trading strategy ϕ^s until the claim is cashed in by the buyer at some stopping time τ . At time τ , the seller would give the buyer Y_τ to payoff the claim and could put the remainder of his wealth $V_\tau(\phi^s) - Y_\tau$ in bond for the time period $(\tau, T]$. This would result in a final value of $(V_\tau(\phi^s) - Y_\tau)(1+r)^{T-\tau}$ which is non-negative and is strictly positive with positive probability (under p or p^*).

To take advantage of a buyer's arbitrage opportunity, an investor could buy one ACC at time zero for C_0 , and invest $-C_0$ according to ϕ^b until the time τ when the buyer cashes in the claim. The buyer would then have $V_\tau(\phi^b) + Y_\tau$ at time τ and could put this in bond for the time period $(\tau, T]$ so that the buyer's final wealth would be $(V_\tau(\phi^b) + Y_\tau)(1+r)^{T-\tau}$ which is non-negative and is strictly positive with positive probability.

The following lemma will be used in showing that an initial price of U_0 for the ACC is arbitrage free.

Lemma 2.3.3 *Let ϕ be a trading strategy in stock and bond with discounted value process*

$$\{V_t^*(\phi) = V_t(\phi)/(1+r)^t, t = 0, 1, \dots, T\}.$$

Then, for any $\tau \in \mathcal{T}_{[0,T]}$,

$$E^{p^*} [V_\tau^*(\phi)] = V_0^*(\phi). \quad (2.56)$$

PROOF. By Lemma 2.2.3, $\{V_t^*(\phi), \mathcal{F}_t, t = 0, 1, \dots, T\}$ is a martingale when p^* is used in place of p . Equation (2.56) then follows from Doob's stopping theorem since τ is a bounded stopping time. $\square$

Theorem 2.3.4 *The unique arbitrage free initial price for the American contingent claim is U_0 .*

PROOF. We first show that the arbitrage free price cannot be anything other than U_0 , i.e., we establish uniqueness of an arbitrage free initial price.

Suppose that $C_0 > U_0$. Then there is an arbitrage opportunity for the seller of the American contingent claim. To see this, let ϕ^s denote the trading strategy in stock and bond corresponding to investing U_0 according to the super hedging strategy ϕ^* and $C_0 - U_0$ in bonds at time zero and holding them until time T . Note that the initial value $V_0(\phi^s) = C_0$ and the value $V_t(\phi^s)$ of ϕ^s at time t is at least U_t for $t = 0, 1, \dots, T$. Then for any stopping time $\tau \in \mathcal{T}_{[0,T]}$,

$$\begin{aligned} V_\tau(\phi^s) - Y_\tau &= V_\tau(\phi^s) + (C_0 - U_0)B_\tau - Y_\tau \\ &\geq U_\tau + (C_0 - U_0)B_\tau - Y_\tau \\ &\geq (C_0 - U_0)B_\tau, \end{aligned} \quad (2.57)$$

where $(C_0 - U_0)B_\tau > 0$. Thus, there is an arbitrage opportunity for a seller of the American contingent claim.

On the other hand, suppose that $C_0 < U_0$. Then there is an arbitrage opportunity for the buyer of the American contingent claim. To see this, let ϕ^b denote the trading strategy corresponding to investing $-U_0$ according to the negative $-\phi^*$ of the super hedging strategy ϕ^* and investing $U_0 - C_0$ in bonds for all time. Furthermore, consider the stopping time τ^* (viewed as the time at which the ACC should be cashed in). Note that $V_t(-\phi^*) = -V_t(\phi^*)$ for $t = 0, 1, \dots, T$, and $V_0(\phi^b) = -C_0$. Then, using the fact that $V_{\tau^*}(\phi^*) = U_{\tau^*} = Y_{\tau^*}$, we have

$$V_{\tau^*}(\phi^b) + Y_{\tau^*} = -V_{\tau^*}(\phi^*) + (U_0 - C_0)B_{\tau^*} + Y_{\tau^*} = (U_0 - C_0)B_{\tau^*}.$$

Since $(U_0 - C_0)B_{\tau^*} > 0$, there is an arbitrage opportunity for a buyer of the American contingent claim.

Finally, suppose that $C_0 = U_0$. We need to show that $C_0 = U_0$ is arbitrage free, i.e., that there exists an arbitrage free initial price. We begin by showing that there is no arbitrage opportunity for a seller of an American contingent claim with $C_0 = U_0$. For a proof by contradiction, suppose that there exists a trading strategy ϕ^s such that $V_0(\phi^s) = U_0$ and for each $\tau \in \mathcal{T}_{[0,T]}$, (2.54) holds. Note that $\tau^* \in \mathcal{T}_{[0,T]}$. Then it follows from (2.54) with $\tau = \tau^*$, that $V_{\tau^*}(\phi^s) - Y_{\tau^*} \geq 0$ and that this inequality holds strictly with positive probability under p and hence under p^* . These same properties continue to hold after multiplying the random variable by the random discount factor $(1+r)^{-\tau^*}$, and so it follows that

$$E^{p^*} [V_{\tau^*}^*(\phi^s) - Y_{\tau^*}^*] > 0. \quad (2.58)$$

On the other hand, by (2.56), $E^{p^*} [V_{\tau^*}^*(\phi^s)] = V_0^*(\phi^s) = U_0^*$, and by (ii) and (iii) in Lemma 2.3.1, $E^{p^*} [Y_{\tau^*}^*] = U_0^*$. Combining these two properties, we obtain $E^{p^*} [V_{\tau^*}^*(\phi^s) - Y_{\tau^*}^*] = 0$, which contradicts (2.58). Therefore, no such ϕ^s exists and consequently there is no arbitrage

opportunity for a seller of the American contingent claim.

Next we show that there is no arbitrage opportunity for a buyer of the American contingent claim with $C_0 = U_0$. For this, suppose that there exists a trading strategy ϕ^b such that $V_0(\phi^b) = -U_0$ and a stopping time $\tau \in \mathcal{T}_{[0,T]}$ such that (2.55) holds. For a contradiction, we will show that $E[V_\tau(\phi^b) + Y_\tau] \leq 0$, or equivalently that $E^{p^*}[V_\tau(\phi^b) + Y_\tau] \leq 0$, which in turn is equivalent to showing that $E^{p^*}[V_\tau^*(\phi^b) + Y_\tau^*] \leq 0$ (since we have assumed that $V_\tau(\phi^b) + Y_\tau \geq 0$). By (2.56), $E^{p^*}[V_\tau^*(\phi^b)] = V_0^*(\phi^b) = -U_0^*$. Moreover, by (ii) in Lemma 2.3.1, $E^{p^*}[Y_\tau^*] \leq U_0^*$. Therefore, $E^{p^*}[V_\tau^*(\phi^b) + Y_\tau^*] \leq 0$, and the desired contradiction is obtained. Consequently there is no arbitrage opportunity for a buyer of the American contingent claim. $\square$

2.4 Exercises

1. Consider a single period CRR model with $S_0 = \$50$, $S_1 = \$100$ or $\$25$, $r = 0.25$.

- Find the arbitrage free initial price of a European call option for one share of stock where the strike price is $K = \$50$ and the exercise time $T = 1$.
- Find a hedging strategy that replicates the value of the option described in (a).
- Suppose the option in (a) is initially priced at \$1 above the arbitrage free price. Describe a strategy (for trading in stock, bond and the option) that is an arbitrage.
- What is the arbitrage free initial price for a put option with the same strike price and exercise time as the call option described in (a)?

2. Consider a CRR model with $T = 2$, $S_0 = \$50$, $S_1 = \$100$ or $S_1 = \$25$, and an associated European call option with strike price $K = \$40$ and exercise time $T = 2$. Assume that the risk free interest rate is $r = 0.1$.

- Draw the binary tree and compute the arbitrage free initial price of the European call option at time zero.
- Determine an explicit hedging strategy for this option.
- Suppose that the option in (a) is initially priced \$2 below the arbitrage free price. Describe a strategy (for trading in stock, bond and the option) that is an arbitrage.
- Try to automate the pricing of a European call option in a computer program where T, S_0, u, d, K are variables.

3. Consider the same CRR model as in Exercise 2, but with an associated European contingent claim whose value at its exercise time $T = 2$ is given by

$$X = \max\{S_0, S_1, S_2\}. \quad (2.59)$$

Answer questions (a)–(d) of Exercise 2 for this contingent claim.

4. Consider the multi-period CRR model and a European contingent claim X . Suppose that trading in the contingent claim is allowed at each of the times $t = 0, 1, \dots, T-1$, where the price of the contingent claim at time t is given by an $\mathcal{F}_t$ -measurable random variable C_t . Let ϕ^* be a replicating strategy (in stock and bond) for the contingent claim X . Show that

$$V_t(\phi^*) = \frac{1}{(1+r)^{T-t}} E^{p^*}[X | \mathcal{F}_t], \quad t = 0, 1, \dots, T. \quad (2.60)$$

Formulate a notion of an arbitrage opportunity in the market consisting of stock, bond and contingent claim. Show that the process $C_t =$

$V_t(\phi^*), t = 0, 1, \dots, T-1$ defines the unique arbitrage free price process for the contingent claim.

5. Consider a CRR model with terminal time T . A European call option with strike price K and expiration date T has value $C_T = (S_T - K)^+$ at time T . A European put option with the same strike price and expiration date has value $P_T = (K - S_T)^+$ at time T . Note that $C_T - P_T = S_T - K$. Let $\{C_t, t = 0, 1, \dots, T-1\}$ and $\{P_t, t = 0, 1, \dots, T-1\}$ denote the arbitrage free price processes for the call and put options, respectively (cf. Exercise 4). Show that the following call-put parity relation holds:

$$C_t - P_t = S_t - K/(1+r)^{T-t}, \quad t = 0, 1, \dots, T-1. \quad (2.61)$$

Note, you can show this either using a “no arbitrage” argument or using the formulas for C_t, P_t resulting from Exercise 3.

6. Consider a CRR model with $T = 2$, $S_0 = \$50$, $S_1 = \$100$ or $S_1 = \$25$ (the same model as in Exercise 2). Now consider an American put option with strike price $K = \$40$. Assume that the risk free interest rate is $r = 0.1$.

- (a) Use a binary tree to compute the arbitrage free initial price of the American put option.
- (b) Determine an explicit super hedging strategy ϕ^* for this option.
- (c) Suppose that you can buy the American put option at time zero for \$1 less than its arbitrage free price. Explicitly describe a strategy that yields an arbitrage opportunity for a buyer of the American put option.
- (d) Try to automate the arbitrage free pricing of an American put option in a computer program where T, S_0, u, d, K are variables.

7. Prove that the arbitrage free initial price for an American call option is the same as the arbitrage free initial price for a European call option. Furthermore, show that with this initial (arbitrage free) price, if the buyer of an American call option waits until the expiration date to exercise the option, this prevents the seller from making a risk free profit (that is, it prevents a seller’s arbitrage).