

Introduction to Mathematical Finance

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Financial Markets and Derivatives

1.1 Financial Markets

A *financial market* consists of tradable securities such as stocks, bonds, currencies, commodities, or even indexes. One reason for the existence of financial markets is that they facilitate the flow of capital. For example, if a company wants to finance the building of a new production facility, it might sell shares of stock to investors who buy the shares based on the anticipation of future rewards, such as dividends or a rise in the stock price.

A variety of stochastic models is used in modeling the prices of securities. All such models face the usual trade off, namely, more complex models typically provide a better fit to data (although there is the danger of overfitting), whereas simpler models are generally more tractable and despite their simplicity can sometimes provide useful qualitative insights. Finding a good balance between a realistic and a tractable model is part of the art of stochastic modeling.

Both discrete (in time and state), and continuous (in time and state) models will be considered here. The treatment of discrete time models is intended as a means of introducing notions such as hedging and pricing by arbitrage in a simple setting and to provide a bridge to

the development in the continuous case. Binomial tree models are the most common example in the discrete case. Our treatment of discrete and continuous models is not meant to be exhaustive, but rather to provide an introduction that will enable students to go on to read more advanced material.

An *arbitrage* opportunity is an opportunity for a risk free profit. A financial market is said to be *viable* if there are no arbitrage opportunities. Typically, liquid financial markets move rapidly to eliminate arbitrage opportunities. Accordingly, in this course, attention will focus on financial market models that are viable.

1.2 Derivatives

A *derivative* or *contingent claim* is a security whose value depends on the value of some underlying security. Examples of derivatives are forward contracts, futures contracts, options, swaps, etc. (We recognize that the reader may not be familiar with all of these terms. Some of them are described in more detail below. For the other terms, or more detailed descriptions in general, we refer the reader to the book by J. Hull [14] or Musiela and Rutkowski [18].) The underlying security on which a derivative is based could be a security in a financial market, such as a stock, bond, currency or commodity, but it could also be a derivative itself, such as a futures contract. Secondary financial markets can be formed from derivatives. The “derivatives market” is huge – a book written in 1996 (by Baxter and Rennie [1]) described it then as a \$15 trillion market, and in the 1999 printing of a book by S. Pliska [19] this market is cited as having \$20 trillion in principal outstanding.

A *forward contract* is an agreement to buy or sell an asset at a certain future time for a certain price. Forward contracts are traded *over*

the counter (OTC), i.e., they are traded between individual entities — typically financial institutions and/or their corporate clients. One of the common uses of forward contracts is to lock in an exchange rate for a future purchase in a foreign currency.

A *futures contract* is similar to a forward contract that is traded on a financial exchange. Futures contracts (or simply futures) differ from forward contracts in a number of respects. In particular, futures typically have a delivery month rather than a delivery date and they follow a settlement procedure called *marking to market*. Briefly, in this procedure, an investor's initial deposit is adjusted on each trading day to reflect gains or losses in the futures price for that day. For further details, we refer the reader to [14] or [18]. Futures contracts are available for a wide range of commodities and financial assets.

An *option* is a contract which gives the holder of the option the right, but not the obligation, to buy or sell a given security at a given price (called the exercise or *strike price*) within a fixed time period $[0, T]$. A *call option* gives the option holder the right to buy at the given price, whereas a *put option* gives the option holder the right to sell at the given price. A *European option* can only be exercised by the holder of the option at the expiration time T , whereas an *American option* can be exercised by the holder at any time in $[0, T]$.

Remark. Options based on stocks are usually written to cover the buying or selling of 100 shares of a stock. This is convenient since shares of stock are usually traded in lots of 100. Most exchange traded options on stock are American-style options. Stock options on the Chicago Board Options Exchange effectively expire on the third Friday of the expiration month.

One might ask, why are options worth buying or selling? To help answer this, consider an example. To keep the description simple, con-

sider a European call option.

Example. On January 4, 2000, a European call option on Cisco (symbol: CSCO) stock has a price of \$33. The option expires in January and the strike price is \$70. The price of Cisco stock on January 4 is \$102.

If one bought such an option on 100 shares of Cisco, the option would cost \$3,300, and on January 21, 2000 (third Friday of January), one would have the right to buy 100 shares of Cisco at a price of \$70 per share. Suppose for simplicity that \$1 on January 4 is worth \$1 on January 21, 2000.

Scenario 1: Suppose the price of Cisco stock on January 21 is \$120 per share. This current price of the stock is called the *spot price* of the stock. The holder of the option will exercise it and make a net profit per share of $\$120 - \$70 - \$33$ (spot price of stock on January 21 — price under exercise of option — option price) and hence a net profit of \$1,700. This is a $\frac{1,700}{33}\% = 51.5\%$ profit on the \$3,300 initial investment. On the other hand, if the \$3,300 had been directly invested in stock, the investor could have bought 32 whole shares of stock and the profit would have been $\$18 \times 32 = \576 , on an investment of $\$102 \times 32 = \$3,264$, which is a $\frac{57600}{3264}\% = 17.6\%$ profit.

Scenario 2: Suppose the price of Cisco stock on January 21 is \$67 per share. The holder of the option will not exercise it and makes a loss of \$33 per share (the cost of the option per share) and hence a net loss of \$3,300. This is a 100% loss on the \$3,300 initial investment. On the other hand, if the \$3,300 had been invested directly in stock, the loss would have been $\$35 \times 32 = \$1,120$ or a 34.3% loss on an investment of \$3,264 in stock.

There are two main uses of derivatives, namely, *speculation and hedging*. For example, the buyer of a call option on a stock is leveraging

his/her investment. The option may not cost much compared to the underlying stock and the owner of the call option can benefit from a rise in the stock price without having to buy the stock. Of course, if the stock price goes down substantially and the option is not exercised, the buyer loses what he/she paid for the call option. In this case, the seller of the call option would benefit, without necessarily having to buy the stock. Of course, if the stock price goes up substantially and the option seller does not already own the stock, then he/she will have to pay the price of buying the stock at the increased price and turning it over to the owner of the option. Options may be used to leverage and speculate in the market. Depending on how the investment is used, the down side losses can be substantial. On the other hand, options can be used to hedge risk. An investor *hedges* when he or she reduces the risk associated with his or her portfolio holdings by trading in the market to reduce exposure of the portfolio to the effect of random fluctuations in market prices. For example, an owner of stock who wants to hedge against a dramatic decrease in the stock price could buy a put option to sell the stock at a certain price, thereby reducing the risk of loss associated with such a dramatic decrease.

The focus of this course will be on the *pricing of derivative securities*. Derivatives will be referred to henceforth as contingent claims. Both European and American contingent claims will be considered here. (In this book, we restrict to contingent claims having a single payoff (cf. [9], rather than allowing the more general situation of [15] where a contingent claim can have an income stream over an interval of time.) A *European contingent claim* has a single expiration date which is the same as its payoff date. The owner of an *American contingent claim* can choose to have the claim payoff at any time up until the expiration date and the payoff may depend on when the owner chooses to cash in the claim. The main principle behind pricing of both types of contingent

claims is that the price should be set in such a way that there is no opportunity for arbitrage in a market where the contingent claim and underlying securities are traded. The existence of an arbitrage free price for a contingent claim hinges on the assumption that there are no opportunities for arbitrage in the underlying (primary) securities market, i.e., that market is viable. Uniqueness of the arbitrage free price depends in the case of a European contingent claim on being able to replicate the payoff of the claim by trading in the (primary) securities market, and in the case of an American contingent claim on the seller of the claim being able to hedge the sale by trading in the market so as to have a portfolio whose value equals or exceeds the payoff of the claim at all times and equals it at some (random) time. These hypotheses will be defined precisely in the context of a model later on, and conditions under which they hold will be given.

The following further assumptions are made to simplify the presentation. There are (a) no transaction costs, and (b) no dividends on stock. It is assumed that (c) the market is liquid, and (d) trading by the investor does not move the market. In the discrete model framework, an investor is allowed (e) unlimited short selling of stock, and (f) unlimited borrowing. In the continuous model framework these assumptions will be modified to prevent “doubling strategies” which result in an arbitrage opportunity.

1.3 Exercises

1. Using a list of option prices (e.g., from the website for the Chicago Board Options Exchange or Yahoo Finance), perform a similar calculation to that done in the Example in the text with Intel (symbol: INTC) in place of Cisco. For this, use the first Intel call option expiring in the current month and for which a price is listed. You should assume that

a European call option for 100 shares of stock is purchased and that at expiration there are two possible scenarios for the stock price – it has gone up or down by 40% since purchase of the option.