

Office hours.

Th. 11-11:45 am or by appointment.

No class on Thu. Oct. 12.

Tu. Oct. 24, Th. Oct. 26

Continuous Mapping Theorem.

Fix d, k integers, positive.

Let $\{h_n\}_{n=1}^{\infty}, h$ be measurable functions from $\mathbb{D}^d$ into $\mathbb{D}^k$.

Define $C_h = \{x \in \mathbb{D}^d : h_n(x_n) \rightarrow h(x) \text{ in } \mathbb{D}^k \text{ whenever } x_n \rightarrow x \text{ in } \mathbb{D}^d\}$.

Let $\{X_n\}_{n=1}^{\infty}, X$ be stochastic processes with paths in $\mathbb{D}^d$,

and suppose $X_n \Rightarrow X$ (in distribution) as $n \rightarrow \infty$.

Assume $P(X \in C_h) = 1$ (a.s. paths of X are in continuity paths of h).

Then $h_n(X_n) \Rightarrow h(X)$ as $n \rightarrow \infty$.

Proof.

By Skorokhod representation theorem,

$\exists \{Y_n\}, Y$ on $(\Omega, \mathcal{F}, P)$, st.

$$Y_n \xrightarrow{\text{dist}} X_n, \forall n$$

$$Y \xrightarrow{\text{dist}} X, \Rightarrow P(Y \in C_h) = 1.$$

and $Y_n \rightarrow Y$ a.s. as $n \rightarrow \infty$.

a.s. ; $Y_n \rightarrow Y$ and $Y \in C_h$.

Consider $\omega \in \Omega : Y_n(\omega) \rightarrow Y(\omega)$ and $Y(\omega) \in C_h$
" " " "
 $X_n \quad X \in C_h$

By definition of C_h , $h_n(X_n(\omega)) \rightarrow h(Y(\omega))$ as $n \rightarrow \infty$

$$\Rightarrow \underset{\text{dist.}}{h_n(X_n)} \rightarrow \underset{\text{dist.}}{h(X)} \text{ as } n \rightarrow \infty \quad (\text{a.s.} \Rightarrow \text{convergence in dist.})$$

$$\Rightarrow h_n(X_n) \Rightarrow h(X) \text{ as } n \rightarrow \infty.$$

Remark.

If $h_n = h$ is a continuous function on $\mathbb{D}^d$ into $\mathbb{D}^k$,

this gives $h(X_n) \Rightarrow h(X)$

Example. $h(x)_t = \sup_{0 \leq s \leq t} |x_s|$, $h: \mathbb{D}^d \rightarrow \mathbb{D}$
 h is continuous.

2. $\{x_n\}$ n.w. $\hat{x}_t^n \Rightarrow B_t$, $|\hat{x}_t^n| \Rightarrow |B_t|$.

Random Time Change Theorem

$\mathbb{D}_0 = \{\varphi \in \mathbb{D} : 0 \leq \varphi(t) < \infty, \forall t, \varphi \text{ is non-decreasing}\}$.

If $x \in \mathbb{D}^d$, $x \circ \phi \in \mathbb{D}^d$

Theorem: Suppose $\{x^n\}$, X are d -diml stochastic processes with paths in $\mathbb{D}^d$,
 $\{\phi^n\}$, Φ are one-diml stochastic processes with paths in $\mathbb{D}_0$.

If $(x^n, \phi^n) \Rightarrow (X, \Phi)$ as $n \rightarrow \infty$,
and $P(X \in \mathbb{C}^d) = 1$

then $x^n \circ \phi^n \Rightarrow X \circ \Phi$ as $n \rightarrow \infty$

$(x^n, \phi^n) \in \mathbb{D}^d \times \mathbb{D} \neq \mathbb{D}^{d+1}$ in topological sense

$\mathbb{D}^d \times \mathbb{D}^d : (x_n, \phi_n) \rightarrow (x, \phi)$
 $\Leftrightarrow x_n \rightarrow x \text{ and } \phi_n \rightarrow \phi$
 $\mathbb{D}^{d+1} : (x_n, \phi_n) \rightarrow (x, \phi)$ need one time change $\{\gamma_n\} : (x_n, \phi_n)(\gamma_n) \rightarrow (x, \phi)$ uniformly on compact time intervals.
 $\mathbb{C}^d \times \mathbb{C} = \mathbb{C}^{d+1}$

If. Use cts. mapping theorem.

prove $\psi: \mathbb{D}^d \times \mathbb{D}_0 \rightarrow \mathbb{D}^d$ defined by

$\psi(x, \phi) = x \circ \phi$ is continuous at $x \in \mathbb{C}^d$.

Functional Central Limit Theorem, (Donsker's theorem)

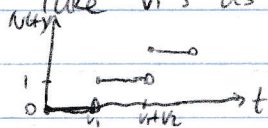
Let $v_1, v_2, \dots$ i.i.d. (real valued) with mean $\mu \in (-\infty, \infty)$,
and variance $\sigma^2 \in (0, \infty)$

Let $\hat{V}^n(t) = \frac{1}{\sigma \sqrt{n}} \sum_{i=1}^{[nt]} (v_i - \mu)$, $t \geq 0$ $[] =$ integer part.

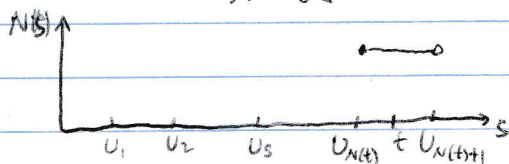
$\hat{V}^n \Rightarrow B = \{B(t), t \geq 0\}$ Brownian motion. Continuous paths,
stationary, independent increments ($B(0) = 0$).

Functional CLT for renewal processes

Take v_i 's as in previous thm, assume $v_i \geq 0 \forall i$, let $\lambda = \frac{1}{\mu}$, $\mu \neq 0$.
renewal process $N(t) = \sup \{k: v_1 + \dots + v_k \leq t\}$.
say Poisson if v_i 's are exponentially distributed.



$$\hat{N}^n(t) := \frac{N(nt) - \lambda nt}{\lambda^{1/2} \sigma \sqrt{n}} \quad \hat{N}^n \Rightarrow B \text{ as } n \rightarrow \infty$$



$$U_m = U(m) = \sum_{i=1}^m v_i \quad U_{N(t)} \leq t \leq U_{N(t)+1}$$

$$\Rightarrow U_{N(t)} \leq nt < U_{N(t)+1}$$

$$\Rightarrow U_{N(t)} - \mu nt \leq nt - \mu nt < U_{N(t)+1} - \mu nt$$

$$\Rightarrow \frac{U_{N(t)} - \mu N(t)}{\sigma \sqrt{n}} \leq \frac{nt - \mu N(t)}{\sigma \sqrt{n}} < \frac{U_{N(t)+1} - \mu N(t)}{\sigma \sqrt{n}}$$

$$- \hat{N}^n(t) \lambda^{1/2} \quad \approx \frac{N(nt) - \lambda nt}{\lambda^{1/2} \sigma \sqrt{n}}$$

$$\frac{U_m - \mu m}{\sigma \sqrt{m/nt}} \Rightarrow \sqrt{\lambda} N(0, t) = \sqrt{\lambda} B_t$$

$$\Rightarrow -\hat{N}^n \Rightarrow B \Rightarrow \hat{N}^n \Rightarrow -B \stackrel{\text{dist}}{=} B$$

Biochemical Reaction Networks

Reference: Anderson + Kurtz, Continuous time Markov chain models for chemical reaction networks. $(S_1, \dots, S_n)$

S : finite set of chemical species. eg. $\{A, B, C\}, \{A_1, A_2, A_3\}$.

G : finite set of complexes consist of everything that is left hand side or right hand side of a reaction.

R : finite set of reactions $i \rightarrow j$

use vectors of length n for record complexes

eg. $V = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ represents $S_1 + S_2$.

$V_k \rightarrow V_k'$ where $V_k, V_k' \in G$

$V_k' - V_k$ is change in #'s of each of species when k^{th} reaction fires.

Reaction example:

