

Topics in Probability

Math 289A

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Topic: Limit Theorems for high dimensional stochastic processes with applications to biological and engineered networks

1. First order approximation (functional law of large numbers): dynamical system; odes
2. Second order approximation (functional central limit theorem): diffusion process (continuous strong Markov process)
3. Stein's method(stationary distribution)
4. Applications

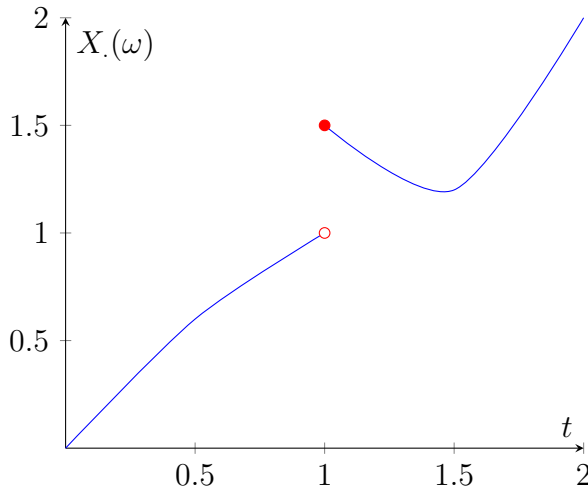
References

1. Billingsley, Convergence of probability measures
2. Ethier + Kurtz, Markov Processes Characterization and Convergence
3. Jacod + Shiryaev, Limit Theorem for Stochastic Processes

Basic Setting $(\Omega, \mathcal{F}, P)$

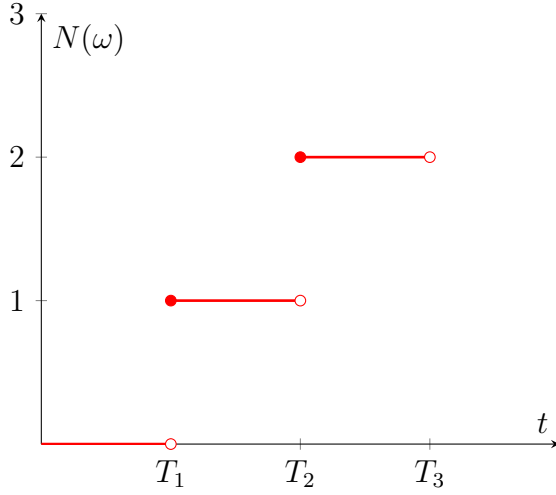
Stochastic Process - Ways to think about them

- (i) "Fix t ": $\{X_t, t \in [0, \infty)\}$ where each X_t is a random variable, i.e., $X_t : \Omega \rightarrow \mathbb{R}^d$ (d - diml Euclidean space) is measurable.
- (ii) "Fix ω ": $X_\cdot(\omega) : [0, \infty) \rightarrow \mathbb{R}^d$ sample path of X . For example,



(All stochastic processes we consider will have sample paths that are at least right continuous with finite left limits) Then $X : \Omega \rightarrow \mathbb{D}([0, \infty), \mathbb{R}^d)$. $X(\omega) = X_\cdot(\omega)$ in notation. We use $\mathbb{C}([0, \infty), \mathbb{R}^d)$ to represent the subset of $\mathbb{D}([0, \infty), \mathbb{R}^d)$ consisting of continuous functions.

Example: Poisson Process



(iii) $X : [0, \infty) \times \Omega \rightarrow \mathbb{R}^d$ and $X(t, \omega) = X_t(\omega)$.

Convergence in function space

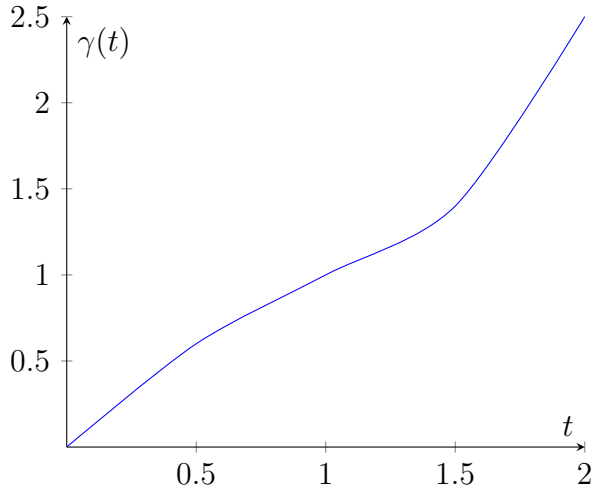
Let $\mathbb{C}^d = \mathbb{C}([0, \infty), \mathbb{R}^d)$. A sequence $\{x_n\}_{n=1}^\infty$ in $\mathbb{C}^d$ converges to x in $\mathbb{C}^d$ if for each fixed $T > 0$, $x_n \rightarrow x$ as $n \rightarrow \infty$ uniformly on $[0, T]$. Define $\|x\|_T = \sup_{t \in [0, T]} |x(t)|$ where $|\cdot|$ is the Euclidean norm on $\mathbb{R}^d$ (w.l.o.g assuming L^2 norm).

So $x_n \rightarrow x$ uniformly on $[0, T] \iff \|x_n - x\|_T \rightarrow 0$ as $n \rightarrow \infty$. Convergence in $\mathbb{C}^d$ means uniform convergence on compact time intervals. Metric on $\mathbb{C}^d$ consistent with this is $d(x, y) = \sum_{m=1}^\infty 2^{-m} (\|x - y\|_m \wedge 1)$. $\mathbb{C}^d$ with this metric is a complete separable metric space (Polish space).

Let $\mathbb{D}^d = \mathbb{D}([0, \infty), \mathbb{R}^d)$: let

$$\Gamma = \{\gamma : [0, \infty) \rightarrow [0, \infty); \gamma \text{ is strictly increasing and continuous, } \gamma(0) = 0, \lim_{t \rightarrow \infty} \gamma(t) = \infty\}.$$

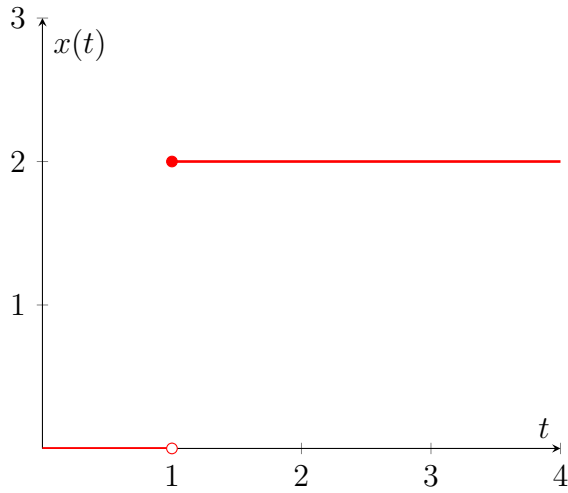
For example,

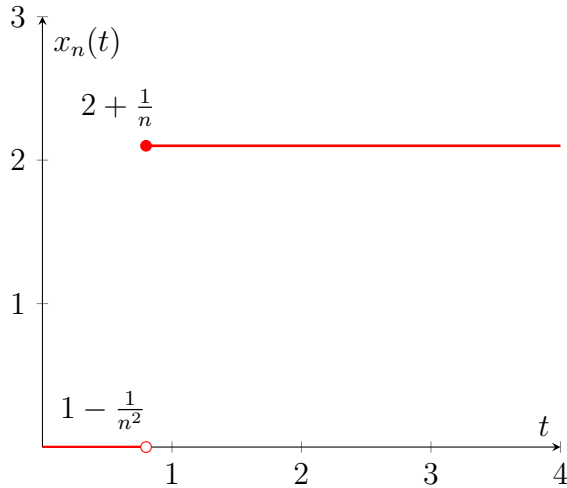


A sequence $\{x_n\}$ in $\mathbb{D}^d$ converges to x in $\mathbb{D}^d$ if for each $T > 0$, there is a sequence $\{\gamma_n\}_{n=1}^\infty$ in Γ such that

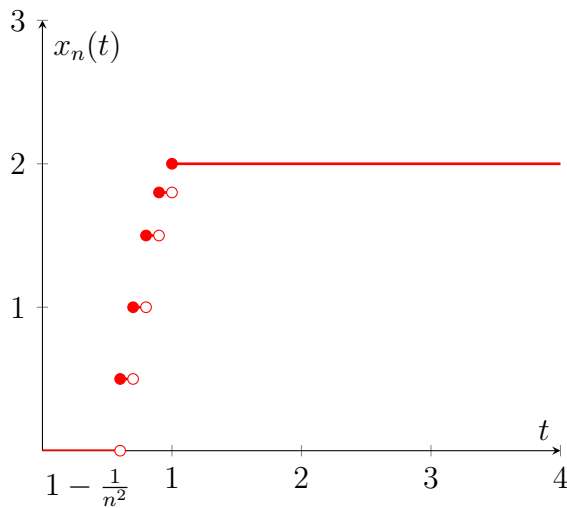
- (i) $\sup_{t \in [0, T]} |\gamma_n(t) - t| \rightarrow 0$ as $n \rightarrow \infty$ and
- (ii) $\sup_{t \in [0, T]} |x_n(\gamma_n(t)) - x(t)| \rightarrow 0$ as $n \rightarrow \infty$.

Example:





From above, we can see it requires the location and size of the jumps in x_n 's to both be close to those of x . Metric associated with this convergence is called the Skorokhod J_1 -metric. There is also an M_1 -metric. The difference is that in M_1 -metric, x_n can have steps to approximate a big jump of x . For example, x_n is as below and x is as above.



Ascoli-Arzelà Theorem

A sequence $\{x_n\}$ in $\mathbb{C}([0, T], \mathbb{R}^d)$ is relatively compact

- (i) there is finite M such that $\|x_n\|_T \leq M < \infty$ for all n , and

(ii) for each $\epsilon > 0$, $\exists \delta > 0$ such that $|x_n(s) - x_n(t)| < \epsilon$ wherever $|s - t| < \delta$ for all n (equicontinuity).

$$\text{Equicontinuity} \iff \sup_n w(x_n, \delta, T) = \sup_n \sup_{s, t \in [0, T], |s-t| < \delta} |x_n(s) - x_n(t)| \rightarrow 0$$

as $\delta \rightarrow 0$ where $w(x_n, \delta, T)$ is called the modulus of continuity of x_n .

For $\mathbb{D}^d$, let

$$w'(x, \delta, T) = \inf_{\text{partition } \{t_i\}_{i=1}^m \text{ of } [0, T]} \max_{i=1}^m \sup_{s, t \in [t_{i-1}, t_i]} |x(s) - x(t)|$$

where the inf is over all finite partitions $\{t_i\}_{i=1}^m$ of $[0, T]$ for which $\min(t_i - t_{i-1}) > \delta$ for $i = 1, \dots, m$. Then $w'(x, \delta, T)$ is called the modified modulus of continuity of x .

Proposition 1 *A set $A \subset \mathbb{D}^d$ is relatively compact (i.e. every sequence has a convergent subsequence) iff for each $T > 0$*

$$(i) \sup_{x \in A} \|x\|_T < \infty$$

$$(ii) \lim_{\delta \rightarrow 0} \sup_{x \in A} w'(x, \delta, T) = 0.$$