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GI/GI/1 fluid model

$$\bar{Q}(t) = \bar{Q}(0) + (\lambda - \mu)t + \mu \bar{I}(t).$$

$\bar{I}(t)$ is cts., non-decr. t can incr. only when $\bar{Q}$ is 0.

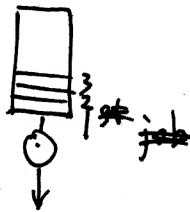
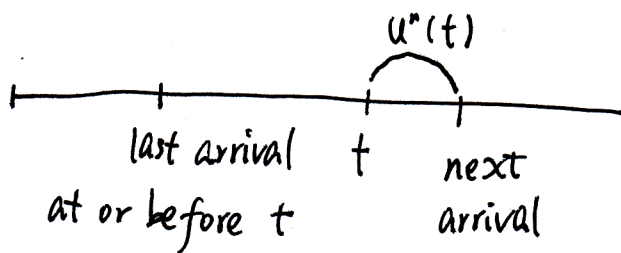
Fluid model is stable if $\exists N > 0$ s.t. for any fluid model sol'n $\bar{Q}$, $\bar{Q}(t) = 0$, $\forall t \geq N|\bar{Q}(0)|$. (Scale by $|\bar{Q}(0)|$.)

Dai's Thm (1995) - see Branson notes.

Impose mild conditions.

Assume interarrival times $\{u_i\}_{i=1}^{\infty}$ have unbounded support, i.e. $\forall K > 0$ $P(u_i > K) > 0$. [Can be arbitrarily large.]

Consider $\chi^n(t) = (U^n(t), V^n(t), \{s_i^n\}_{i=1}^{\infty})$.



[Keep track of residual interarrival time $U^n(t)$, residual

service time for job in service $V^n(t)$, $\{s_i^n\}_{i=1}^{\infty}$ the "age" of i th job in the queue. The first job is in the head of the line.]

We make the sequence infinite. If no i th job, make $s_i = -1$.

Want to know how old the jobs are if we've more than 1 queue.

Recover queue length: $Q^n(t) = \sum_{i=1}^{+\infty} 1_{\{S_i^n \geq 0\}}$.

Want to have $\{S_i^n\}$ in some deterministic space. So make its length infinite. $\mathbb{R}_+ \times \mathbb{R}_+ \times (\mathbb{R}_+ \cup \{-1\})^{\mathbb{N}}$ Markov state descriptor for GI/GI/1 queue.

Suppose fluid model is stable, then χ^n is positive Harris recurrent for any n . [Harris is from USC.]

Now have continuous state! Harris Recurrent: $\sup_{x \in A} E_x[\tau_A^n(\delta)] < \infty$, where

$\tau_A^n(\delta) = \inf \{t \geq \delta : \chi^n(t) \in A\}$, $\delta > 0$, continuous analog of "first return time", A is some "petite" set.

Defn. A petite set A is a non-empty, measurable set for which $\exists$ a prob. measure a on $(0, +\infty)$ & a non-trivial pos. measure ν , s.t. $\nu(B) \leq \int_0^{+\infty} p_t(x, B) a(dt)$ $\forall x \in A$ & B meas. sets. $\square$

$$[p_t^n(x, B) = P_x(\chi^n(t) \in B).]$$



$p_t^*(x, \cdot)$ is bounded below in some sense by some reference measure ν . Continuous analog of communicating property.

The converse of Dai's Thm is generally not true. Dai's Thm is useful in proving stability.

GI/GI/1 diffusion approx.

Consider $\lambda = \mu$ (balanced).

(Can also consider $n(\lambda^n - \mu^n) \rightarrow \theta$ as $n \rightarrow \infty$.)

$$Q^n(t) = Q^n(0) + A(t) - S(T^n(t)).$$

$$\begin{aligned} \hat{Q}^n(t) &= \frac{Q^n(n^2 t)}{n} = \hat{Q}^n(0) + \frac{A(n^2 t) - \lambda n^2 t}{n} \\ &\quad - \frac{S(T^n(n^2 t)) - \mu T^n(n^2 t)}{n} + \frac{\lambda n^2 t - \mu n^2 t}{n} + \frac{\mu n^2 t - \mu T^n(n^2 t)}{n} \\ &= \hat{Q}^n(0) + \hat{A}^n(t) - \hat{S}^n(\bar{T}^n(t)) + \mu \hat{I}^n(t). \end{aligned}$$

$$\begin{aligned} [\hat{A}^n(t) &= \frac{A(n^2 t) - \lambda n^2 t}{n}, \quad \hat{S}^n(t) = \frac{S(n^2 t) - \mu n^2 t}{n}, \quad \bar{T}^n(t) \\ &= \frac{T^n(n^2 t)}{n^2}. \quad T^n(n^2 t) = \int_0^{n^2 t} 1_{\{Q^n(s) > 0\}} ds \neq \frac{n^2 t - T^n(n^2 t)}{n} \\ &= \frac{1}{n} \int_0^{n^2 t} 1_{\{Q^n(s) = 0\}} ds = \hat{I}^n(t) = \frac{I^n(n^2 t)}{n} \text{ non-decr. + incr. only} \end{aligned}$$

when $\hat{Q}^n = 0$.]

$$\tilde{S} = \frac{S}{n^2}. \quad \hat{I}^n(t) = n \int_0^t |\{Q^n(n^2 \tilde{S}) = 0\}| d\tilde{S} = n \int_0^t |\{\hat{Q}^n(\tilde{S}) = 0\}| d\tilde{S}.$$

Claim. $\hat{I}^n(t) = \mu^{-1} \Phi(\hat{Q}^n(0) + \hat{A}^n(\cdot) - \hat{S}^n(\bar{\bar{T}}^n(\cdot)))(t) \quad \forall t \geq 0$,

where $\Phi(x)(t) = \sup_{0 \leq s \leq t} x^-(s)$, $x^-(s) = \max(-x(s), 0)$.

0). $\Phi: \mathbb{D} \rightarrow \mathbb{D}$ (cts. mapping). $\square$

Suppose $\hat{Q}^n(0) \Rightarrow \tilde{Q}(0)$ & $\hat{Q}^n(0)$ ~~is~~ indept. of $\hat{A}^n, \hat{S}^n$.

FCLT $\Rightarrow (\hat{A}^n, \hat{S}^n) \Rightarrow (B_a, B_s)$.

B_a is 1-dim'l BM (variance par. $\lambda^3 \sigma_a^2$) and B_s is 1-dim'l BM (variance par. $\mu^3 \sigma_s^2$). B_a, B_s indept.

$\{\bar{\bar{T}}^n\}_{n=1}^\infty$ are C-tight and any weak limit pt. $\bar{Q}(t) =$

$$\bar{Q}(0) + \lambda t - \mu \bar{\bar{T}}(t). \quad \bar{Q}^n(t) = \frac{\hat{Q}^n(t)}{n} \Rightarrow \bar{Q}(0) = 0.$$

$$\bar{Q}(t) = \lambda t - \mu \bar{\bar{T}}(t) = (\lambda - \mu)t + \mu \bar{\bar{I}}(t) = \mu \bar{\bar{I}}(t).$$

For balanced fluid model, $\bar{Q}(t) = \bar{Q}(0) \quad \forall t$.

$$\Rightarrow \bar{\bar{I}}(t) = 0 \quad \forall t. \Rightarrow \bar{\bar{T}}(t) = t \quad \forall t.$$

So conclude $\bar{\bar{T}}^n \Rightarrow \bar{\bar{T}}(t) = t$, all t .

Summa

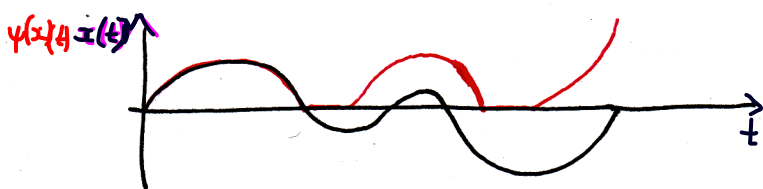
Summary

$(\hat{Q}^n(0), \hat{A}^n, \hat{S}^n, \bar{\bar{T}}^n) \Rightarrow (\tilde{Q}(0), B_a, B_s, \bar{\bar{T}})$ as $n \rightarrow \infty$. For $\hat{X}^n(t) = \hat{Q}^n(0) + \hat{A}^n(t) - \hat{S}^n(\bar{\bar{T}}^n(t))$, $\hat{X}^n \Rightarrow \tilde{X}$ where $\tilde{X}(t) = \tilde{Q}(0) + B_a(t) - B_s(t)$. BM var. par. $\lambda^3 \sigma_a^2 + \mu^3 \sigma_s^2$.

$$\hat{Q}^n(t) = \hat{X}^n(t) + \Phi(\hat{X}^n)(t) = \psi(\hat{X}^n)(t).$$

$$\psi(x)(t) = x(t) + \Phi(x)(t).$$

$\psi: \mathbb{D} \rightarrow \mathbb{D}$ is cts. By cts. mapping thm, $\psi(\hat{X}^n) \Rightarrow \psi(\tilde{X}) \Rightarrow \tilde{Q}^n \Rightarrow \tilde{Q}$, where $\tilde{Q}(t) = \tilde{X}(t) + \Phi(\tilde{X})(t)$. $\tilde{X}$ BM, $\Phi(\tilde{X})$ cts., non-decr. + incr. only when $\tilde{Q} = 0$. $\tilde{Q}$ is 1-dim'l reflected BM.



$$(\hat{Q}^n, \hat{I}^n) \Rightarrow (\tilde{Q}, \tilde{I}), \quad \tilde{I} = \Phi(\tilde{X}). \text{ Local time of } \tilde{Q}.$$