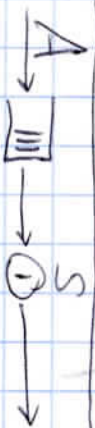


G1/G1/1 FIFO



Sequence of systems, indexed by n :

$$Q^n(t) = Q^n(0) + A(t) - S(T^n(t)), \quad T^n(t) = \int_0^t \mathbf{1}_{\{Q^n(s) > 0\}} ds$$

$$I^n(t) = t - T^n(t)$$

$$W^n(t) = W^n(0) + \sum_{i=1}^{A(t)} v_i - T^n(t)$$

Service times associated with arrivals

$$\text{Let } V(t) = \sum_{i=1}^t v_i, \quad V(1) = \sum_{i=1}^{[t]} v_i$$

FCLT:

$$\left\{ \frac{V(n+1) - \frac{1}{\mu} \cdot n+1}{\sigma \sqrt{n+1}}, t \geq 0 \right\} \Rightarrow \mathcal{B}$$

Standard Gaussian motion

Fluid scaling:

$$\text{e.g. } \bar{Q}^n(t) = \frac{Q^n(n+1)}{n}$$

Fluid scaled equations:

$$\bar{Q}^n(t) = \bar{Q}^n(0) + \bar{A}^n(t) - \bar{S}^n(T^n(t)) \leftarrow \text{Assume } \bar{Q}^n(0) \Rightarrow \mathcal{B}(0)$$

$$\bar{I}^n(t) = t - \bar{T}^n(t)$$

$$\bar{W}^n(t) = \bar{W}^n(0) + \bar{V}^n(\bar{A}^n(t)) - \bar{T}^n(t)$$

Recall FLLN:

$$(\bar{A}^n, \bar{S}^n) \Rightarrow (\lambda(\cdot), \mu(\cdot))$$

①

$\{A^n, S^n\}_{n=1}^\infty$ is ~~not~~ C-tight

$\{T^n\}_{n=1}^\infty$ is C-tight $\Leftarrow$ subsequence $T^{n_k} \Rightarrow T$

$\{V^n\}_{n=1}^\infty$ is C-tight

By random time change then,

$\{\bar{S}^n(T^n(\cdot))\}_{n=1}^\infty$ is C-tight

Follows that $\{(Q^n, T^n, \bar{W}^n)\}_{n=1}^\infty$ is C-tight.

Then $\{(A^n, S^n, T^n, V^n, Q^n, T^n, \bar{W}^n)\}_{n=1}^\infty$ is C-tight.

$\rightarrow$ wait out:

• If ~~$X^n \rightarrow X$~~ $X^n \rightarrow X$, $Y^n \rightarrow Y$ in $\mathbb{C}^d$,

then $(X^n, Y^n) \rightarrow (X, Y)$ in $\mathbb{C}^{2d}$.

• If $X^n \rightarrow X$, $Y^n \rightarrow Y$ in $\mathbb{D}^d$,

then $(X^n, Y^n) \not\rightarrow (X, Y)$ in $\mathbb{D}^{2d}$.

• Recall $X^n \xrightarrow{a.s.} X$, $Y^n \xrightarrow{a.s.} Y \not\Rightarrow (X^n, Y^n) \xrightarrow{a.s.} (X, Y)$

• $\mathbb{D}^{2d} \neq \mathbb{D} \times \mathbb{D}$, but $\mathbb{C}^{2d} = \mathbb{C} \times \mathbb{C}$

Suppose (A, S, T, V, Q, T, W) is a weak fluid limit point

limit point of $\{(A^n, S^n, T^n, V^n, Q^n, T^n, W^n)\}$.

then random time change theorem

$$\begin{aligned} Q(t) &= Q(0) + A(t) - S(T(t)) \\ T(t) &= t - T(t) \end{aligned}$$

$$W(t) = W(0) + \frac{1}{n} A(t) - T(t)$$

Fluid model equations

②

Properties:

(1) $\bar{T}$ inherits unif.

Lip. cont. from $\{\bar{T}^n\}_{n=1}^\infty$

$\Rightarrow \bar{T}$ is differentiable

almost everywhere:

$$\bar{T}(t) = \int_0^t \bar{T}^*(s) ds$$

define to be "0" where $\bar{T}^*$ is not d.f.

$$\Rightarrow \bar{Q}(t) = \bar{Q}(0) + \lambda t - \mu \bar{T}(t)$$

$\Rightarrow \bar{Q}(t)$ is also absolutely continuous

$$\text{for a.e. } t > 0, \quad \dot{\bar{Q}}(t) = \lambda - \mu \bar{T}^*(t)$$

$$\text{where } \bar{I}(t) = \lim_{k \rightarrow \infty} \bar{T}^n(t) = \lim_{k \rightarrow \infty} \int_0^t \frac{1}{k} \bar{Q}^n(s) ds$$

Claim: $\bar{T}$ can only increase where $\bar{Q}$ is zero



~~use~~ use SR thus:

$$\Rightarrow \bar{I}(t) = \lim_{k \rightarrow \infty} \int_0^t \frac{1}{k} \bar{Q}^n(s) ds \leq \lim_{k \rightarrow \infty} \int_0^t \frac{1}{k} \bar{Q}^n(s) ds$$

$\bar{Q}^n \rightarrow \bar{Q}$ unif. on compact time intervals a.s. (u.a.c., a.s.)

$$\Rightarrow f_Q(\bar{Q}^n) \rightarrow f_Q(\bar{Q}) \text{ u.a.c. a.s.} \quad (3)$$

(1):

$$\bar{T}^n(t) = \int_0^t \frac{1}{k} \bar{Q}^n(s) ds$$

unif. Lip. continuous:

$$|\bar{T}^n(t) - \bar{T}^n(s)| \leq |t - s|$$

$$\Rightarrow |\bar{T}(t) - \bar{T}(s)|$$

$$= \lim_{k \rightarrow \infty} |\bar{T}^n(t) - \bar{T}^n(s)| \leq |t - s|$$

$$\Rightarrow \bar{I}(t) \leq \lim_{t \rightarrow \infty} \int_0^t f(\bar{Q}(s)) ds$$

$$\int_0^t f(\bar{Q}(s)) ds$$

↓ as $t \rightarrow \infty$

$$\int_0^t 1_{\{\bar{Q}(s) = 0\}} ds \rightarrow \int_0^t 1_{\{\bar{Q}(s) = 0\}} ds$$

Similarly, for $0 \leq s < t < \infty$,

$$\bar{I}(t) - \bar{I}(s) \leq \int_s^t 1_{\{\bar{Q}(u) = 0\}} du \quad \checkmark \text{ (claim)}$$

hence

Recall: For a.e. $t > 0$,

$$\dot{\bar{Q}}(t) = \begin{cases} \lambda - \mu + "0", & \text{if } \bar{Q}(t) > 0 \text{ (} \dot{\bar{I}}(t) = 0 \text{)} \\ 0, & \text{if } \bar{Q}(t) = 0 \end{cases}$$

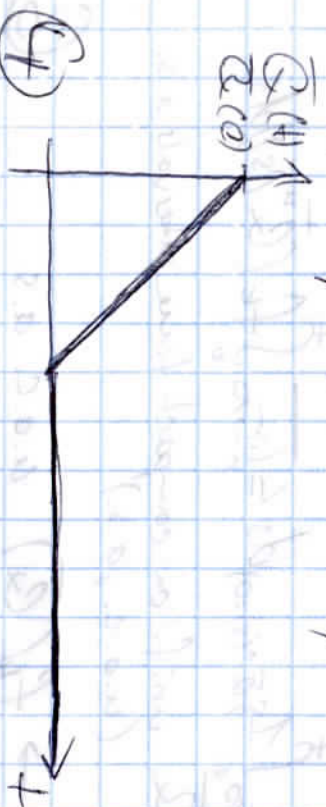


$$\text{Derivative from left: } \dot{\bar{Q}}(t) = \lim_{h \downarrow 0} \frac{\bar{Q}(t) - \bar{Q}(t-h)}{h} \leq 0$$

$$\text{Derivative from right: } \dot{\bar{Q}}(t) = \lim_{h \downarrow 0} \frac{\bar{Q}(t+h) - \bar{Q}(t)}{h} \geq 0$$

$$\Rightarrow \bar{Q}(t) = 0 \text{ whenever } \bar{Q} \text{ differentiable at } t \text{ and } \bar{Q}(t) = 0$$

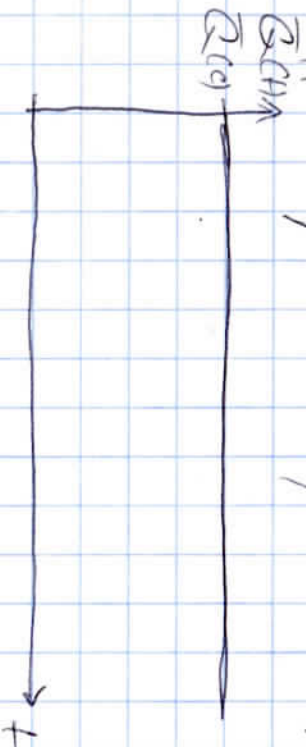
If $\lambda < \mu$ (underloaded/subcritical queue):



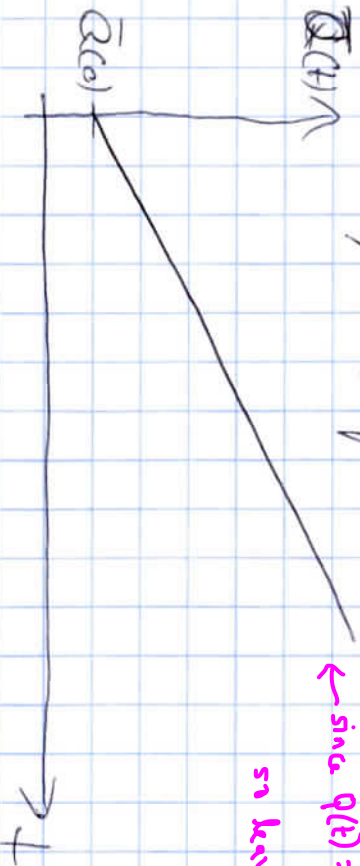
If $\bar{I}(t) = \int_0^t \mathbb{1}_{\{\bar{Q}(s) > 0\}} ds$, $\bar{Q} = 1 - \mu + \mu$ if $\bar{Q} = 0$
 $\uparrow$
 not true!
 $= \lambda \mu \bar{Q} = 0$

("→ contradiction")

If $\lambda = \mu$ (critically loaded, heavy traffic):



If $\lambda > \mu$ (supercritical):



← since $\bar{Q}(t) = \bar{Q}(0) + (\lambda - \mu)t + \mu \bar{I}(t)$
 $\geq \bar{Q}(0) + (\lambda - \mu)t$
 so leaves 0 instantly if $\bar{Q}(0) = 0$

Definition: ("uniformly locally")

Fluid model is stable if $\exists N > 0$ s.t.

$Q(t) = 0 \quad \forall t \geq N Q(0)$ for any fluid model solution.