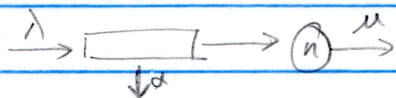


Stein's method.

$M|M|n+M$ (Erlang-A)



Define $R = \frac{\lambda}{\mu}$ $\delta = \frac{1}{\sqrt{R}}$

$X(t)$ = # customers in system at time t

$$= X(0) + N_1(\lambda t) - N_2\left(\int_0^t \alpha (X(s) - n)^+ ds\right) - N_3\left(\int_0^t \mu (X(s) \wedge n) ds\right)$$

"Fluid model"

Suppose $X = \tilde{\lambda} r$, $n = \tilde{n} r$

$$\bar{X}^r(t) = \frac{X^r(t)}{r} = \frac{X^r(0) + N_1(\tilde{\lambda} r t) - N_2\left(\int_0^t \alpha (r \bar{X}^r(s) - r \tilde{n}) ds\right) - N_3\left(\int_0^t \mu (r \bar{X}^r(s) \wedge r \tilde{n}) ds\right)}{r}$$

$$= \frac{1}{r} \left(\right.$$

$$O\left(\frac{1}{\sqrt{r}}\right)$$

$$\frac{N_1(\lambda r t)}{r} = \frac{\tilde{\lambda} r t}{r} + \left[\frac{W(r \tilde{\lambda} t)}{r} \right] + \frac{O(\log(r \tilde{\lambda} t))}{r}$$

$$\bar{X}^r(0) + \tilde{\lambda} t - \alpha \int_0^t (\bar{X}^r(s) - \tilde{n})^+ ds - \mu \int_0^t (\bar{X}^r(s) \wedge \tilde{n}) ds + O\left(\frac{1}{\sqrt{r}}\right)$$

$r \rightarrow +\infty$ The limit of $\bar{X}^r$ satisfies

$$X(t) = X_0 + \tilde{\lambda} t - \alpha \int_0^t (X(s) - \tilde{n})^+ ds - \mu \int_0^t (X(s) \wedge \tilde{n}) ds.$$

Multiply by r :

$$rX(t) = rX_0 + r\lambda t - \alpha \int_0^t (rX(s) - rn)^+ ds - \mu \int_0^t (rX(s) \wedge rn) ds$$

$$\Rightarrow \bar{X}(t) = \bar{X}(0) + \lambda t - \alpha \int_0^t (\bar{X}(s) - n)^+ ds - \mu \int_0^t (\bar{X}(s) \wedge n) ds$$

continuous GDE approximation.

Fluid model for X .

Hoping: $X(t) \approx \bar{X}(t)$, $t \geq 0$

$t \rightarrow \infty$

$$\boxed{X(\infty) - \bar{X}(\infty)}$$

$$\sqrt{R} \left(\frac{X^R(\infty) - \bar{X}^R(\infty)}{R} \right) \approx Y$$

The Fluid model has a unique fixed point (steady state) $\bar{X}(\infty)$ (where $\dot{\bar{X}} = 0$)

i.e.

$$\lambda = \alpha (\bar{X}(\infty) - n)^+ + \mu (\bar{X}(\infty) \wedge n)$$

Two cases:

if $R \geq n$, $(\lambda \geq n\mu)$ server always busy

$$\bar{X}(\infty) = \frac{\lambda - n\mu}{\alpha} + n$$

$R < n$ $(\lambda < n\mu)$ $\lambda = 0 + \mu \bar{X}(\infty) \Rightarrow \bar{X}(\infty) = \frac{\lambda}{\mu} = R$

Let

$$\tilde{X}(t) = (X(t) - \bar{x}(\infty)) \delta$$

$$\zeta = \delta (x(\infty) - n)$$

$$\begin{aligned} b(x) &= \delta (\lambda - \mu (R(x) \wedge n) - \alpha (R(x) - n)^+) \quad R(x) = \frac{x}{\delta} + \bar{x}(\infty) \quad x \in \mathbb{R} \\ &= [\zeta + x]^- - [\zeta]^- \mu - [\zeta + x]^+ - [\zeta]^+ \alpha \end{aligned}$$

X is positive recurrent, so $\tilde{X}$ is also positive recurrent

let $\tilde{X}(\infty)$ be a r.v. with stationary distribution for $\tilde{X}(\cdot)$
(i.e. $\tilde{X}(t) \Rightarrow \tilde{X}(\infty)$ as $t \rightarrow \infty$)

let $\tilde{Y}(\infty)$ be a r.v. with density given by $k \exp(-\frac{1}{\mu} \int_0^x b(y) dy)$, $x \in \mathbb{R}$
 k : normalizing const

Wasserstein distance (

$$d_W(\tilde{X}(\infty), \tilde{Y}(\infty)) = \sup_{h \in \text{Lip}(1)} |E[h(\tilde{X}(\infty))] - E[h(\tilde{Y}(\infty))]|$$

Theorem:

If $d_w(U_m, V) \rightarrow 0$ as $m \rightarrow \infty$

then $U_m \Rightarrow V$ as $m \rightarrow \infty$

Convergence in W-distance implies convergence in distribution

Kolmogorov distance:

$$d_K(\tilde{X}(\infty), \tilde{Y}(\infty)) = \sup_{h \in \mathcal{H}_K} |E[h(\tilde{X}(\infty))] - E[h(\tilde{Y}(\infty))]|$$

$\parallel$
 $\{1_{(-\infty, a]} : a \in \mathbb{R}\}$

If V has bdd density function (bound is 1)

$$\text{Then } d_K(U, V) \leq \sqrt{2cd_w(U, V)}$$

Theorem 2. in paper

Consider the Erlang-A system ($\alpha > 0$).

There exists an increasing function $C_w: \mathbb{R}_+ \rightarrow \mathbb{R}_+$

s.t. for all $n \geq 1$, $\lambda > 0$, $\mu > 0$, $\alpha > 0$.

and $R \triangleq \lambda/\mu \geq 1$

$$d_w(\tilde{X}(\infty), \tilde{Y}(\infty)) \leq \delta C_w\left(\frac{\alpha}{\mu}\right) \quad (R \rightarrow \infty, \delta \rightarrow 0, d_w \rightarrow 0)$$

Theorem 4

Erlang-A ($\alpha > 0$)

There exists an increasing function $C_k: \mathbb{R}_+ \rightarrow \mathbb{R}_+$, s.t.

$\forall n \geq 1, \lambda > 0, \mu > 0, \alpha > 0, R \geq 1$

$$d_k(\hat{X}(\infty), \hat{Y}(\infty)) \leq \delta C_k\left(\frac{\alpha}{\mu}\right)$$

Infinitesimal generator

Let $G_{\tilde{X}}$ be the i.g. for $\tilde{X}$ $G_{\tilde{X}}$ is operator on nice f .

(for nice f)

$$(G_{\tilde{X}} \cdot f)(x) := \lim_{t \rightarrow 0} \frac{(P_t f) - f(x)}{t} = \lim_{t \rightarrow 0} \frac{E[f(\tilde{X}_t) | \tilde{X}_0 = x] - f(x)}{t}$$

$$\text{Let } \tilde{Y}(t) = \tilde{Y}(0) + \underbrace{\int_0^t b(\tilde{Y}(s)) ds}_{\text{drift term}} + B(\sqrt{2\mu}t)$$

$\tilde{Y}$ is a diffusion process.

$\tilde{Y}(\infty)$ has the stationary distribution for $\tilde{Y}$

$$\tilde{Y}(t) \Rightarrow \tilde{Y}(\infty) \text{ as } t \rightarrow \infty$$

$$\tilde{X}(t)$$

$$\Downarrow t \rightarrow \infty$$

$$\tilde{X}(\infty)$$

$$\tilde{Y}(t)$$

$$\Downarrow$$

$$\tilde{Y}(\infty)$$

Infinitesimal generator for $\tilde{Y}(\cdot)$:

$$f \in C^2(\mathbb{R})$$

$$G_{\tilde{Y}} f(x) = b(x)f'(x) + \mu f''(x)$$

By Ito's formula

$$\begin{aligned} f(\tilde{Y}(t)) &= f(\tilde{Y}(0)) + \int_0^t f'(\tilde{Y}(s)) b(\tilde{Y}(s)) ds \\ &\quad + \int_0^t f'(\tilde{Y}(s)) dB(\sqrt{\mu} s) \\ &\quad + \frac{1}{2} \int_0^t f''(\tilde{Y}(s)) 2\mu ds. \end{aligned}$$

$$\Rightarrow E[f(\tilde{Y}(t))] = E[f(\tilde{Y}(0))] + E\left[\int_0^t G_{\tilde{Y}} f(\tilde{Y}(s)) ds\right]$$

$$(P_t f)(y) = f(y) + E\left[\int_0^t (G_{\tilde{Y}} f)(\tilde{Y}(s)) ds\right]$$

$$\begin{aligned} \Rightarrow \frac{P_t f(y) - f(y)}{t} &= \frac{1}{t} E\left[\int_0^t (G_{\tilde{Y}} f)(\tilde{Y}(s)) ds\right] \\ &\rightarrow (G_{\tilde{Y}} f)(y) \text{ as } t \rightarrow \infty. \end{aligned}$$

Let $\mathcal{H} = \text{Lip}(1)$ ($\forall h \in \mathcal{H} : |h(x) - h(y)| \leq |x - y| \quad \forall x, y$)

Suppose that for $h \in \mathcal{H}$ f_h satisfies

$$(G_{\tilde{Y}} f)(x) = E[h(\tilde{Y}(\infty))] - h(x) \quad \forall x$$

Poisson's
Equation

Replace x with $\tilde{x}(\infty)$

$$E[G_{\tilde{Y}} f_h(\tilde{x}(\infty))] = E[h(\tilde{Y}(\infty))] - E[h(\tilde{x}(\infty))]$$

$$\text{thus } |E(G_{\tilde{Y}} f_h(\tilde{x}(\infty)))| = |E[h(\tilde{Y}(\infty))] - E[h(\tilde{x}(\infty))]|$$