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9.1 One dimensional Skorokhod Problem

As mentioned before,

$$\mathbb{D} = \{x : [0, \infty) \rightarrow \mathbb{R}, x \text{ is right continuous and has finite left limits}\}$$

denotes the space where paths of the processes that we consider lie.

$$\mathbb{D}_+ = \{x \in \mathbb{D}, x(0) \geq 0\}$$

denotes paths in $\mathbb{D}$ starting in the positive orthant.

The Skorokhod Problem (denoted by SP in the following) is as defined as follows:

Def. Skorokhod Problem. Let $x \in \mathbb{D}_+$, a pair $(z, y) \in \mathbb{D}_+ \times \mathbb{D}_+$ is a solution of the Skorokhod problem for x , if:

(i). $z(t) = x(t) + y(t), \forall t \geq 0.$

(ii). $z(t) \geq 0, \forall t \geq 0.$

(iii). y satisfies:

(a): $y(0) = 0.$

(b): y is non-decreasing.

(c): $\int_{(0, \infty)} z(t) dy(t) = 0.$

Lemma 9.1 Suppose (z, y) satisfies (i), (ii) and (iii)(a),(b). Then (iii)(c) is equivalent to:
 $\forall t \geq 0 :$

1. $z(t) = 0$ whenever $\Delta y(t) = y(t) - y(t-) > 0.$

2. y is flat whenever $z \neq 0.$

Note: These two conditions imply that y can be decomposed into a continuous function plus a pure jump function, where jumps happen only when z is at zero.

Theorem 9.2 Let $x \in \mathbb{D}_+$, then there exists a unique pair $(z, y) \in \mathbb{D}_+ \times \mathbb{D}_+$ solving the SP for x .
 In fact,

$$y(t) = \max_{0 \leq s \leq t} x^-(s)$$

where

$$x^-(s) = (-x(s)) \vee 0$$

is the unique solution.

Furthermore, if x has continuous paths, then y and z have continuous paths.

Proof:

1. Uniqueness:

Suppose (y, z) and $(\tilde{y}, \tilde{z})$ are two solutions. Then
$$\begin{cases} z = x + y \\ \tilde{z} = x + \tilde{y} \end{cases}$$

WTS: $f = z - \tilde{z} = y - \tilde{y} = 0$,

First of all, $f(0) = y(0) - \tilde{y}(0) = 0$.

Notice that y & $\tilde{y}$ are non-decreasing, therefore f is locally of bounded variation.

Proposition 9.3 (Folland, Chap 3.)

Suppose F and G are right continuous with finite left limits and of bounded variation, then

$$F(b)G(b) - F(a-)G(a-) = \int_{[a,b]} \frac{F(x) + F(x-)}{2} dG(x) + \int_{[a,b]} \frac{G(x) + G(x-)}{2} dF(x)$$

Take $F = G = f$, since $f(0) = 0$ and we define $f(0-) = 0$, we have:

$$\begin{aligned} f^2(t) - f^2(0) &= f^2(t) \\ &= 2 \int_{(0,t]} f(s) df(s) - \sum_{0 < s \leq t} [\Delta f(s)]^2 \end{aligned}$$

Therefore:

$$\begin{aligned} f^2(t) + \sum_{0 < s \leq t} (\Delta f(s))^2 &= 2 \int_{(0,t]} f(s) df(s) \\ &= 2 \int_{(0,t]} (z - \tilde{z})(s) d(y - \tilde{y})(s) \\ &= 2 \left[\int_{(0,t]} z(s) dy(s) + \int_{(0,t]} \tilde{z}(s) d\tilde{y}(s) - \int_{(0,t]} z(s) d\tilde{y}(s) - \int_{(0,t]} \tilde{z}(s) dy(s) \right] \end{aligned}$$

Where the first two terms equal 0 by definition of the (SP), and the last two terms are less than or equal to 0 because $z, \tilde{z}$ are non-negative and $y, \tilde{y}$ are non-decreasing. Therefore we have The LHS of this equation is

non-positive. But the LHS is non-negative by its form, therefore it equals 0.

2. Existence:

It's easy to check that

$$y(t) = \max_{0 \leq s \leq t} x^-(s)$$

satisfies properties (iii)(a) and (iii)(b). To show (iii)(c), i.e.

$$\int_{[0, \infty)} z(t) dy(t) = 0,$$

first, consider $t > 0$ such that $\Delta y(t) > 0$. Then, since $\Delta y(t) = y(t) - y(t-)$, $y(t) > y(t-)$ and so $x^-(t) > \sup_{s \in [0, t)} x^-(s) \geq 0$. It follows that $y(t)$ must equal $x^-(t) = -x(t)$ and so $z(t) = x(t) + y(t) = 0$. Thus we have proved that

$$\Delta y(t) > 0 \text{ implies that } z(t) = 0.$$

Now suppose that $0 < s \leq t$ and $y(t) > y(s-)$. We aim to show that $z(u) = 0$ for some $u \in [s, t]$. If $\Delta y(u) > 0$ for some $u \in [s, t]$, then we have $z(u) = 0$ by the relation we just proved. On the other hand, if y is continuous on $[s, t]$ and $y(s) = y(s-)$, then we need to prove that $\exists u \in [s, t], y(u) = -x(u)$, from which we have $z(u) = x(u) + y(u) = 0$.

For this, let $u^* = \inf\{u > s : y(u) > y(s)\}$, then $u^* \in [s, t]$ since $y(t) - y(s) > 0$, and we have $y(u) = y(s) \forall u \in [s, u^*]$, since y is assumed to be continuous on $[s, t]$. From the cumulative nature of y , $\forall u \geq u^*$, $y(u) = y(u^*) \vee \sup\{-x(v) : v \in [u^*, u]\}$. Combining this with the definition of u^* , we see that $\forall \epsilon > 0, \exists v_\epsilon \in [u^*, u^* + \epsilon]$, such that $y(s) = y(u^*) < -x(v_\epsilon)$. Letting $\epsilon \rightarrow 0$, we obtain by right continuity of x that $y(u^*) \leq -x(u^*)$. But since $y(u^*) \geq -x(u^*)$ by definition, we have $y(u^*) = -x(u^*)$. ■

9.1.1 How to use the solution to the Skorokhod Problem

Suppose $B(\cdot)$ is a Brownian motion. Fix $\omega \in \Omega$, let $x(t) = x_0 + B(t, \omega)$, where $x_0 \geq 0$. Since Brownian motion has continuous paths, $x \in \mathbb{C}_+$. If we find the solution (z, y) to the SP of x , then we have

$$z(t, \omega) = x_0 + B(t, \omega) + \max_{s \leq t} (-(x_0 + B(s, \omega)), 0),$$

z is the reflected path of x and the resulting process is a (one-dimensional) reflected Brownian motion.

If we let $x_0 = 0$, then

$$z(\cdot) \stackrel{d}{=} |B(\cdot)|.$$

In fact, **Tanaka's formula** tells the following:

$$z(t) = |B(t)| = \int_0^t \text{sgn}(B(s)) dB(s) + L_t,$$

where

$$L_t = \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t 1_{(-\varepsilon, +\varepsilon)}(B(s)) ds.$$

9.1.2 Lipschitz property of the Skorokhod mapping

Given $x \in \mathbb{D}_+$, with (z, y) it's SP solution. Define $\phi(x) = z$, $\psi(x) = y$, then ϕ, ψ are continuous w.r.t. the J_1 -topology on the space $\mathbb{D}$.

Moreover, for $x, \tilde{x} \in \mathbb{D}_+$, we have:

$$\|\phi(x) - \phi(\tilde{x})\|_t = \max_{s \leq t} |\phi(x)(s) - \phi(\tilde{x})(s)| \leq 2 \|x - \tilde{x}\|_t,$$

$$\left\| \psi(x) - \psi(\tilde{x}) \right\|_t \leq \|x - \tilde{x}\|_t.$$

References

[NOTES] J.G. DAI and R.J. WILLIAMS, "Skorokhod problem."