

Topics in Probability (Math 289A): Lecture 7

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Functional Central Limit Theorem

Recall

$$\hat{V}^r(t) = \sqrt{r}(\bar{X}^r(t) - \bar{X}(t)),$$

$$\hat{V}^r(t) = \hat{V}^r(0) + \sum_{k=1}^K (v'_k - v_k) [\tilde{W}_k(\int_0^t f_k^r(\bar{X}^r(s))ds) + \int_0^t \nabla f_k^r(\rho_k^r(s)) \cdot \hat{V}^r(s)ds] + \zeta^r(t),$$

and $\zeta^r \Rightarrow 0$ as $r \rightarrow \infty$, $\rho_k^r(s) \in [\bar{X}^r(s), \bar{X}(s)]$, $\tilde{W}_k(\int_0^t f_k^r(\bar{X}^r(s))ds) \Rightarrow \tilde{W}_k(\int_0^t f_k(\bar{X}(s))ds)$.

Need to show $\{\hat{V}^r\}_{r>1}$ is C -tight.

Suppose $\hat{V}^r \Rightarrow \hat{V}$. It was shown last time that

$$\hat{V}(t) = \hat{V}(0) + \sum_{k=1}^K (v'_k - v_k) [\int_0^t \nabla f_k(\bar{X}(s)) \cdot \hat{V}(s)ds + \int_0^t \sqrt{f_k(\bar{X}(s))} d\tilde{W}_k(s)]. \quad (1)$$

Assume $\hat{V}^r(0) \Rightarrow \hat{V}(0)$. Let $G(t) = \sum_{k=1}^K (v'_k - v_k) \int_0^t \sqrt{f_k(\bar{X}(s))} d\tilde{W}_k(s)$; $H(s) = \sum_{k=1}^K (v'_k - v_k) \nabla f_k(\bar{X}(s))$. Then we can rewrite (1) as

$$\hat{V}(t) = \hat{V}(0) + G(t) + \int_0^t H(s) \hat{V}(s)ds. \quad (2)$$

Given $\hat{V}(0)$, there is a unique continuous solution $\hat{V}(t)$ of (2) (we solve the equation ω by ω).

On the righthand side of the equation, G and H are known but $\hat{V}$ is unknown.

Let $U(t) = \int_0^t H(s) \hat{V}(s)ds$, $J(t) = H(t) \cdot \hat{V}(0) + H(t) \cdot G(t)$, then we have

$$\frac{dU}{dt} = H(t) \cdot \hat{V}(t) = J(t) + H(t) \cdot U(t)$$

$$\begin{aligned}
\implies \frac{d(e^{-\int_0^t H(s)ds}U(t))}{dt} &= e^{-\int_0^t H(s)ds} \frac{dU}{dt} - e^{-\int_0^t H(s)ds} H(t)U(t) \\
&= e^{-\int_0^t H(s)ds} \left[\frac{dU}{dt} - H(t)U(t) \right] \\
&= e^{-\int_0^t H(s)ds} J(t).
\end{aligned}$$

Integrate:

$$\begin{aligned}
e^{-\int_0^t H(s)ds}U(t) &= \int_0^t e^{-\int_0^s H(u)du} J(s)ds \\
\implies U(t) &= e^{\int_0^t H(s)ds} \int_0^t e^{-\int_0^s H(u)du} J(s)ds.
\end{aligned} \tag{3}$$

Now we prove the C -tightness of $\{\hat{V}^r\}_{r>1}$.

Since $\{\bar{X}^r\}$ satisfies compact containment and ∇f_k is continuous on Euclidean space, then $\forall \epsilon > 0, \exists r^{**}, K^{**}$ such that $P(\|\nabla f_k^r(\rho_k^r(\cdot))\|_T \leq K^{**}) \geq 1 - \epsilon, \forall r \geq r^{**}$. And since we know $\{\hat{V}^r(0), \sum_{k=1}^K (v'_k - v_k) \tilde{W}_k(\int_0^\cdot f_k^r(\bar{X}^r(s))ds), \zeta^r(\cdot)\}$ is C -tight, then for each $T > 0, \epsilon > 0, \exists K^\dagger, r^\dagger$ such that $\forall r \geq r^\dagger$,

$$\begin{aligned}
P\left(\|\hat{V}^r(0) + \sum_{k=1}^K (v'_k - v_k) \tilde{W}_k\left(\int_0^\cdot f_k^r(\bar{X}^r(s))ds\right) + \zeta^r(\cdot)\|_T \leq K^\dagger, \right. \\
\left. \text{and } \|\nabla f_k^r(\rho_k^r(\cdot))\|_T \leq \frac{K^\dagger}{(|v'_k - v_k| + 1)K}, k = 1, 2, \dots, K\right) \geq 1 - \epsilon.
\end{aligned}$$

Denote the “good set” above by $G^{r,\epsilon}$, then on $G^{r,\epsilon}$, for $t \in [0, T]$

$$\begin{aligned}
|\hat{V}^r(t)| &\leq K^\dagger + \sum_{k=1}^K |v'_k - v_k| \int_0^t |\nabla f_k(\rho_k^r(s))| |\hat{V}^r(s)| ds \\
&\leq K^\dagger + K^\dagger \int_0^t |\hat{V}^r(s)| ds \\
&\stackrel{\text{Gronwall}}{\leq} K^\dagger + (K^\dagger)^2 \int_0^t e^{K^\dagger(t-s)} ds \\
&\leq K^\dagger
\end{aligned}$$

$$\begin{aligned}
\implies P(\|\hat{V}^r\|_T \leq K^\dagger) &\geq P(G^{r,\epsilon}) \geq 1 - \epsilon \text{ for } r \geq r^\dagger, \\
\implies \{\hat{V}^r\} &\text{ satisfies compact containment.}
\end{aligned}$$

Compact containment of $\{\nabla f_k^r(\rho_k^r(\cdot))\}$ and $\{\hat{V}^r\}$ implies compact containment for $\{\int_0^\cdot \nabla f_k^r(\rho_k^r(s)) \cdot \hat{V}^r(s)ds\}_{r>1}$.

To control oscillations, consider $\epsilon > 0$, we can make $P(w(\int_0^\cdot \nabla f_k^r(\rho_k^r(s)) \cdot \hat{V}^r(s)ds, \delta, T) \leq \epsilon)$ as close to 1 as we like by choosing δ sufficient small and r sufficient large.

Since $\hat{V}^r(\cdot) = \sqrt{r}(\bar{X}^r(\cdot) - \bar{X}(\cdot)) \Rightarrow \hat{V}(\cdot)$, we have $\bar{X}^r(\cdot) \simeq \bar{X}(\cdot) + \frac{1}{\sqrt{r}}\hat{V}(\cdot)$. ($r \propto \text{volume}$) This is called the LINEAR NOISE APPROXIMATION. This approximation is locally good but not always globally. For example, if the fluid model has two stable fixed points, then the approximation is good near each of the fixed points, but not good for describing the transition of the Markov chain from a well containing one fixed point to a well containing the other. We will discuss some cases of this next time.

By (1) we have

$$d\hat{V}(t) = \sum_{k=1}^K (v'_k - v_k) [\nabla f_k(\bar{X}(t)) \cdot \hat{V}(t) dt + \sqrt{f_k(\bar{X}(t))} d\tilde{W}_k(t)].$$

Grönwall's Inequality Suppose f, g are Lebesgue measurable functions on $[0, T]$ such that

$$f(t) \leq g(t) + c \int_0^t f(s) ds \quad \forall t \in [0, T], \quad (4)$$

where c is a positive constant. Then

$$f(t) \leq g(t) + c \int_0^t e^{c(t-s)} g(s) ds \quad \forall t \in [0, T]. \quad (5)$$

Proof Let $U(t) = \int_0^t f(s) ds$, then for a.e. t ,

$$\begin{aligned} \frac{dU}{dt} &= f(t) \leq g(t) + cU(t) \\ \implies \frac{dU}{dt} - cU(t) &\leq g(t) \\ \implies e^{-ct} \frac{dU}{dt} - ce^{-ct} U(t) &\leq g(t) e^{-ct} \\ \implies \frac{d(Ue^{-ct})}{dt} &\leq g(t) e^{-ct}. \end{aligned}$$

Then by doing integration we get for all $t \geq 0$,

$$\begin{aligned} U(t) e^{-ct} &\leq \int_0^t g(s) e^{-cs} ds \\ \implies U(t) &\leq \int_0^t g(s) e^{c(t-s)} ds. \end{aligned}$$

Substitute in (4) we get $f(t) \leq g(t) + c \int_0^t e^{c(t-s)} g(s) ds$.