

MATH 289 Oct.19 Notes, Lecture 6

Yichi Zhang

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Last time: $\{(\bar{X}^r, \bar{N}_1^r, \dots, \bar{N}_K^r, \bar{\Phi}_1^r, \dots, \bar{\Phi}_K^r)\}$ is C-tight.

Suppose $(\bar{X}^r, \mathbb{1}(\cdot), \bar{\Phi}_1, \dots, \bar{\Phi}_K)$ is a limit (in distribution) obtained along a subsequence indexed by $\{r_k\}_{k=1}^\infty$.

By Skorokhod's representation theorem, can choose distributionally equivalent representation such that the convergence is a.s. u.o.c. (uniform on compact time intervals).

Abusing notation, using the same notation for the distributional representatives, we have $\{(\bar{X}^{r_k}, \bar{N}_1^{r_k}, \dots, \bar{N}_K^{r_k}, \bar{\Phi}_1^{r_k}, \dots, \bar{\Phi}_K^{r_k})\}_{k=1}^\infty$ converges a.s. to $(\bar{X}, \mathbb{1}(\cdot), \bar{\Phi}_1, \dots, \bar{\Phi}_K)$.

Recall: $\bar{X}^{r_k}(t) = \bar{X}^r(0) + \sum_{l=1}^K (v'_l - v_l) \bar{N}_l^{r_k}(\bar{\Phi}_l^{r_k}(t))$

We have a.s.,

- $\bar{X}^{r_k} \rightarrow \bar{X}$ uniformly on $[0, t]$
- $\bar{X}^r(0) \rightarrow x_0$ (assumption)
- $\bar{\Phi}_l^{r_k}(t) \rightarrow \int_0^t f_l(\bar{X}(s)) ds$ since $\bar{\Phi}_l^{r_k}(t) = \int_0^t f_l^{r_k}(\bar{X}^{r_k}(s)) ds$, where $f_l^{r_k}(\bar{X}^{r_k}(\cdot)) \rightarrow f_l(\bar{X}(\cdot))$ uniformly on $[0, t]$ because
 - $\bar{X}^{r_k} \xrightarrow{a.s.} \bar{X}$ uniformly on $[0, t]$
 - $f_l^{r_k} \xrightarrow{a.s.} f_l$ uniformly on compact sets as $k \rightarrow \infty$ ($f_l^{r_k} = f_l + O(\frac{1}{r})$)

$$\begin{aligned} & |f_l^{r_k}(\bar{X}^{r_k}(s)) - f_l(\bar{X}(s))| \\ & \leq |f_l^{r_k}(\bar{X}^{r_k}(s)) - f_l(\bar{X}^{r_k}(s))| + |f_l(\bar{X}^{r_k}(s)) - f_l(\bar{X}(s))| \end{aligned}$$

Thus $\bar{X}^{r_k}(t) \xrightarrow{a.s.} x_0 + \sum_{k=1}^K (v'_l - v_l) \int_0^t f_l(\bar{X}(s)) ds$ for each $t \geq 0$

So a.s., $\bar{X}$ satisfies the ordinary differential equation:

$$\frac{d\bar{X}}{dt} = \sum_{l=1}^K (v'_l - v_l) f_l(\bar{X}(t)) = S \begin{bmatrix} f_1(\bar{X}(t)) \\ \vdots \\ f_K(\bar{X}(t)) \end{bmatrix}$$

$$\bar{X}(0) = x_0$$

where $f_l(x) = \prod_{i=1}^n x_i^{v_{il}}$ and $S = [v'_1 - v_1 \quad \dots \quad v'_K - v_K]$

Since f_l 's are locally Lipschitz continuous, there is a unique solution to this ODE, given x_0 .

Follows from this uniqueness plus C-tightness that the whole sequence converges in distribution, and in particular, $\{\bar{X}^r\}$ converges to $\bar{X}$ in distribution, the unique solution of the ODE, this is a functional law of large numbers. (Hydrodynamic limit. Limit is often called a fluid model solution)

Example: $S_1 + S_2 \xrightleftharpoons[\tilde{\kappa}_2]{\tilde{\kappa}_1} S_3$

$$\bar{X}(\cdot) = x(\cdot), S = \begin{bmatrix} -1 & 1 \\ -1 & 1 \\ 1 & -1 \end{bmatrix}, f_1(x) = \tilde{\kappa}_1 x_1 x_2, f_2(x) = \tilde{\kappa}_2 x_3$$

$$\begin{aligned} \frac{dx_1}{dt} &= -\tilde{\kappa}_1 x_1 x_2 + \tilde{\kappa}_2 x_3 \\ \frac{dx_2}{dt} &= -\tilde{\kappa}_1 x_1 x_2 + \tilde{\kappa}_2 x_3 \\ \frac{dx_3}{dt} &= \tilde{\kappa}_1 x_1 x_2 - \tilde{\kappa}_2 x_3 \end{aligned}$$

Next study fluctuations of $\bar{X}^r$ around $\bar{X}$:

$$\hat{V}^r(t) := \sqrt{r}(\bar{X}^r(t) - \bar{X}(t))$$

Want to study behavior of $\hat{V}^r$ as $r \rightarrow \infty$.

$$\left(\text{If } \hat{V}^r \Rightarrow \hat{V} \text{ then informally } \bar{X}^r(t) = \bar{X}(t) + \frac{1}{\sqrt{r}} \hat{V}^r(t) \simeq \bar{X}(t) + \frac{1}{\sqrt{r}} \hat{V}(t) \right)$$

Recall:

$$\begin{aligned} \bar{X}^r(t) &= \bar{X}^r(0) + \sum_{k=1}^K (v'_k - v_k) \bar{N}_k^r \left(\int_0^t f_k^r(\bar{X}^r(s)) ds \right) \\ &= \bar{X}^r(0) + \sum_{k=1}^K (v'_k - v_k) \left[\int_0^t f_k^r(\bar{X}^r(s)) ds + \frac{1}{\sqrt{r}} \hat{N}_k^r \left(\int_0^t f_k^r(\bar{X}^r(s)) ds \right) \right] \\ \left(\bar{N}^r(t) &= t + \frac{1}{\sqrt{r}} \hat{N}_k^r(t), \hat{N}_k^r(t) = \sqrt{r} (\bar{N}^r(t) - t) \right) \end{aligned}$$

$$\bar{X}(t) = \bar{X}(0) + \sum_{k=1}^K (v'_k - v_k) \int_0^t f_k(\bar{X}(s)) ds$$

Thus

$$\begin{aligned} \hat{V}^r(t) = \hat{V}^r(0) + \sum_{k=1}^K (v'_k - v_k) & \left[\hat{N}_k^r \left(\int_0^t f_k^r(\bar{X}^r(s)) ds \right) + \sqrt{r} \left(\int_0^t f_k^r(\bar{X}^r(s)) ds - \int_0^t f_k(\bar{X}^r(s)) ds \right) \right. \\ & \left. + \sqrt{r} \left(\int_0^t f_k(\bar{X}^r(s)) ds - \int_0^t f_k(\bar{X}(s)) ds \right) \right] \end{aligned}$$

Let

$$\begin{aligned} \text{I} &= \sqrt{r} \left(\int_0^t f_k^r(\bar{X}^r(s)) ds - \int_0^t f_k(\bar{X}^r(s)) ds \right) \\ \text{II} &= \sqrt{r} \left(\int_0^t f_k(\bar{X}^r(s)) ds - \int_0^t f_k(\bar{X}(s)) ds \right) \\ \text{III} &= \hat{N}_k^r \left(\int_0^t f_k^r(\bar{X}^r(s)) ds \right) \end{aligned}$$

If $\bar{X}^r \rightarrow \bar{X}$ a.s., u.o.c.,

- $|\text{II}| = \left| \sqrt{r} \left(\int_0^t f_k(\bar{X}^r(s)) ds - \int_0^t f_k(\bar{X}(s)) ds \right) \right| \leq \frac{C(\omega)\sqrt{rt}}{r} \xrightarrow{r \rightarrow \infty} 0$
Thus $\text{II} \Rightarrow 0$ as $r \rightarrow \infty$

- $\exists \tilde{x}$ between x and y s.t. $f_k(y) = f_k(x) + \nabla f_k(\tilde{x}) \cdot (y - x)$, $\nabla f_k = \begin{bmatrix} \frac{\partial f_k}{\partial x_1} \\ \vdots \\ \frac{\partial f_k}{\partial x_n} \end{bmatrix}$

This is a consequence of the fundamental theorem of calculus and the mean value theorem:

$$\begin{aligned} f_k(y) &= f_k(x) + \int_0^1 \frac{\partial f_k(x + t(y - x))}{\partial t} dt \\ &= f_k(x) + \nabla f_k(x + t(y - x)) \cdot (y - x) \text{ for some } t \in [0, 1] \end{aligned}$$

Let $y = \bar{X}^r(s)$ and $x = \bar{X}(s)$ then

$$\begin{aligned} \text{II} &= \sqrt{r} \int_0^t \nabla f_k(\rho_k^r(s)) \cdot (\bar{X}^r(s) - \bar{X}(s)) ds \text{ for some } \rho_k^r(s) \text{ between } \bar{X}^r(s) \text{ and } \bar{X}(s) \\ &= \int_0^t \nabla f_k(\rho_k^r(s)) \cdot \hat{V}^r(s) ds \end{aligned}$$

- $N_k(u) = u + W_k(u) + l(u)\xi_k(u)$
where $l(u) = (\log u) \vee 2$, $\sup_{u \geq 0} |\xi_k(u)| = \eta_k < \infty$ (strong approximation)

$$\frac{W_k(r\cdot)}{\sqrt{r}} = \tilde{W}_k(\cdot) \text{ (another Brownian motion)}$$

Thus

$$\begin{aligned}\hat{N}_k^r(t) &= \sqrt{r} (\bar{N}_k^r(t) - t) \\ &= \sqrt{r} \left(\frac{N_k(rt)}{r} - t \right) \\ &= \sqrt{r} \left(\frac{rt + W_k(rt) + l(rt)\xi_k(rt)}{r} - t \right) \\ &= \frac{W_k(rt)}{\sqrt{r}} + \frac{l(rt)\xi_k(rt)}{\sqrt{r}} \\ &= \tilde{W}_k(t) + \frac{(\log rt) \vee 2}{\sqrt{r}} \cdot \xi_k(rt)\end{aligned}$$

$$\begin{aligned}\text{so III} &= \tilde{W}_k \left(\int_0^t f_k^r(\bar{X}^r(s)) ds \right) + \frac{(\log(r \int_0^t f_k^r(\bar{X}^r(s)) ds))^{\vee 2}}{\sqrt{r}} \cdot \xi_k \left(r \int_0^t f_k^r(\bar{X}^r(s)) ds \right), \\ \text{where } &\frac{(\log(r \int_0^t f_k^r(\bar{X}^r(s)) ds))^{\vee 2}}{\sqrt{r}} \cdot \xi_k \left(r \int_0^t f_k^r(\bar{X}^r(s)) ds \right) \xrightarrow{r \rightarrow \infty} 0(\cdot) \\ &\text{because}\end{aligned}$$

- $\int_0^t f_k^r(\bar{X}^r(s)) ds \Rightarrow \int_0^t f_k(\bar{X}(s)) ds$
- $\xi_k \left(r \int_0^t f_k^r(\bar{X}^r(s)) ds \right) \leq \eta_k$

Hence

$$\hat{V}^r(t) = \hat{V}^r(0) + \sum_{k=1}^K (v'_k - v_k) \left[\tilde{W}_k \left(\int_0^t f_k^r(\bar{X}^r(s)) ds \right) + \sqrt{r} \int_0^t \nabla f_k(\rho_k^r(s)) \cdot \hat{V}^r(s) ds \right] + \zeta^r(t)$$

where $\int_0^t f_k^r(\bar{X}^r(s)) ds \Rightarrow \int_0^t f_k(\bar{X}(s)) ds$ and $\zeta^r(\cdot) \Rightarrow 0$ as $r \rightarrow \infty$.

Want to show:

1. $\{\hat{V}^r\}$ is C-tight.
2. Assuming 1., get $\{(\hat{V}^r, \bar{X}^r)\}$ is C-tight.

If $(\hat{V}, \bar{X})$ is a limit point, can assume w.l.o.g. that convergence is a.s.,

u.o.c.. Then we obtain from the above that

$$\begin{aligned}
\hat{V}(t) &= \hat{V}(0) + \sum_{k=1}^K (v'_k - v_k) \left[\tilde{W}_k \left(\int_0^t f_k(\bar{X}(s)) ds \right) + \int_0^t \nabla f_k(\bar{X}(s)) \cdot \hat{V}(s) ds \right] \\
&= \hat{V}(0) + \sum_{k=1}^K (v'_k - v_k) \left[\int_0^t \sqrt{f_k(\bar{X}(s))} d\tilde{\tilde{W}}_k(s) + \int_0^t \nabla f_k(\bar{X}(s)) \cdot \hat{V}(s) ds \right] \\
d\hat{V}(t) &= \sum_{k=1}^K (v'_k - v_k) \left[\nabla f_k(\bar{X}(t)) \cdot \hat{V}(t) dt + \sqrt{f_k(\bar{X}(t))} d\tilde{\tilde{W}}_k(t) \right]
\end{aligned}$$