

1 (BIO)CHEMICAL REACTION NETWORK MODEL

Let $\mathcal{S} = \{S_1, S_1, \dots, S_n\}$ be the set of **species**, $\mathcal{C}$ be the set of **complexes**, and $\mathcal{R} = \{v_k \rightarrow v'_k : k = 1, \dots, K\}$ be the set of **reactions**.

Example 1. Consider the reaction $2H + O \rightarrow H_2O$. The species are H, O and H_2O . The complexes are $2H + O$ and H_2O . We write $v = (2, 1, 0)^T$ and $v' = (0, 0, 1)^T$ to represent the complexes $2H + O$ and H_2O respectively.

For each $i = 1, 2, \dots, n$, let $X_i(t)$ be the number of molecule of species S_i in the system at time t . The vector $X(t) = (X_1(t), \dots, X_n(t))^T$ is called the **state descriptor**. The changes in X are generated by reactions: when the k -th reaction $v_k \rightarrow v'_k$ fires, $X(t)$ changes to $X(t) - v_k + v'_k$.

1.1 Simplest Model

In this model X is a continuous time Markov chain, where $X(t)$ takes values in $\mathbb{Z}_+^n$. Suppose $X(t) = x$, each reaction $v_k \rightarrow v'_k$ fires at rate $\lambda_k(x)$.

Theorem 2. (Poisson Process Representation - Kurtz) The process X satisfies

$$X(t) = X(0) + \sum_{k=1}^K (v'_k - v_k) N_k \left(\int_0^t \lambda_k(X(s)) ds \right), \quad (1)$$

where $N_1, \dots, N_K$ are independent Poisson processes with rate 1.

We will assume that λ_k satisfies **mass action kinetics**: $\lambda_k(X(t))$ is proportional to the amount of copies of each of the reacting species. If $v_k = (v_{1k}, v_{2k}, \dots, v_{nk})^T$, we set

$$\lambda_k(x) = \kappa_k \left[\prod_{i=1}^n x_i(x_i - 1) \dots (x_i - v_{ik} + 1) \right] = \kappa_k \left[\prod_{i=1}^n v_{ik}! \binom{x_i}{v_{ik}} \right]. \quad (2)$$

Let r be the volume of the container multiplied by Avogadro's number. We will allow κ_k to be depend on r , and will denote this by κ_k^r . Both λ_k and X are now depend on r , and will be denoted by κ_k^r and X^r respectively.

Example 3. Consider the reaction $S_1 + S_2 \rightarrow S_3$. The rate is given by $\lambda^r(x) = \kappa^r x_1 x_2$, where x_1 and x_2 are the numbers of S_1 and S_2 in the system. If the molecules of S_1 and S_2 have radius r_1 and r_2 , then they will collide if their centers are less than $r_1 + r_2$ apart. The probability that an S_1 molecule and an S_2 molecule collide in a small time interval is proportional to $(\pi(r_1 + r_2)^2/r)(x_1 x_2 / 2!) \propto x_1 x_2 / r$. Hence, we expect $\kappa^r \propto 1/r$.

We will set $\kappa_k^r = \tilde{\kappa}_k / r^{m_k - 1}$, where $m_k = \sum_{i=1}^n v_{ik}$. Let $\bar{X}^r = X^r / r$, which records the concentration of each species. From (2), we have

$$\begin{aligned} \lambda_k^r(x) &= \frac{\tilde{\kappa}_k}{r^{m_k - 1}} \left[\prod_{i=1}^n x_i(x_i - 1) \dots (x_i - v_{ik} + 1) \right] \\ &= r \tilde{\kappa}_k \left[\prod_{i=1}^n \frac{x_i}{r} \left(\frac{x_i}{r} - \frac{1}{r} \right) \dots \left(\frac{x_i}{r} - \frac{v_{ik} - 1}{r} \right) \right] \\ &= r \tilde{\kappa}_k \left[\prod_{i=1}^n \bar{x}_i \left(\bar{x}_i - \frac{1}{r} \right) \dots \left(\bar{x}_i - \frac{v_{ik} - 1}{r} \right) \right] \end{aligned} \quad (3)$$

$$= r f_k^r(\bar{x}^r). \quad (4)$$

From (3), for large r , we have

$$\lambda_k^r(x) \simeq r\tilde{\kappa}_k \prod_{i=1}^n (\bar{x}_i^r)^{v_{ik}} = r\tilde{\kappa}_k (\bar{x}^r)^{v_k}. \quad (5)$$

From (1) and (4), we have

$$\bar{X}^r(t) = \bar{X}^r(0) + \sum_{k=1}^K (v'_k - v_k) \cdot \frac{1}{r} N_k \left(r \int_0^t f_k^r(\bar{X}^r(s)) ds \right). \quad (6)$$

Theorem 4. (*Strong Approximation - Komlos, Major, Tusnady (1975)*) *There is a probability space $(\Omega, \mathcal{F}, P)$ on which a unit rate Poisson process N and a standard one-dimensional Brownian motion W are both defined such that*

$$\eta = \sup_{t \geq 0} \frac{|N(t) - t - W(t)|}{(\log t) \vee 2} < \infty \quad \text{a.s.}, \quad (7)$$

and

$$E[e^{\alpha\eta}] < \infty \quad \text{for some } \alpha > 0. \quad (8)$$

Assume $N_1, \dots, N_K$ are independent Poisson processes, and $W_1, \dots, W_K$ are independent Brownian motions, such that for all k ,

$$\eta_k = \sup_{t \geq 0} \frac{|N_k(t) - t - W_k(t)|}{(\log t) \vee 2} < \infty \quad \text{a.s.} \quad (9)$$

This implies that $N_k(t) = t + W_k(t) + \xi_k(t)$, where $|\xi_k(t)| \leq ((\log t) \vee 2)\eta_k$. From (6), it follows that

$$X^r(t) = \bar{X}^r(0) + \sum_{k=1}^K (v'_k - v_k) \left[\int_0^t f_k^r(\bar{X}^r(s)) ds + \frac{1}{r} W_k \left(r \int_0^t f_k^r(\bar{X}^r(s)) ds \right) + \frac{1}{r} \xi_k \left(r \int_0^t f_k^r(\bar{X}^r(s)) ds \right) \right]. \quad (10)$$

Since $W_k(r\cdot)/\sqrt{r} \stackrel{d}{=} \tilde{W}_k(\cdot)$, we have

$$\frac{1}{r} W_k \left(r \int_0^t f_k^r(\bar{X}^r(s)) ds \right) \stackrel{d}{=} \frac{1}{\sqrt{r}} \tilde{W}_k \left(\int_0^t f_k^r(\bar{X}^r(s)) ds \right). \quad (11)$$

Also,

$$\left| \frac{1}{r} \xi_k \left(r \int_0^t f_k^r(\bar{X}^r(s)) ds \right) \right| \leq \frac{1}{r} \left(\log \left(r \int_0^t f_k^r(\bar{X}^r(s)) ds \right) \vee 2 \right) \eta_k. \quad (12)$$

The R.H.S. of both (11) and (12) seem to go to 0 as r goes to infinity. This leads us to a conjecture that $\bar{X}^r \Rightarrow \bar{X}$, where $\bar{X}$ satisfies

$$\bar{X}(t) = \bar{X}(0) + \sum_{k=1}^K (v'_k - v_k) \int_0^t \lambda_k(\bar{X}(s)) ds. \quad (13)$$

This can also be written in the form of the differential equation $\dot{\bar{X}}(t) = S\lambda(\bar{X}(t))$, where S is a $n \times K$ matrix, where the i -th column is $v'_i - v_i$. This last equation is known as the **reaction rate equation**, and the matrix S is known as the **stoichiometric matrix**.