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math185 - Dropbox

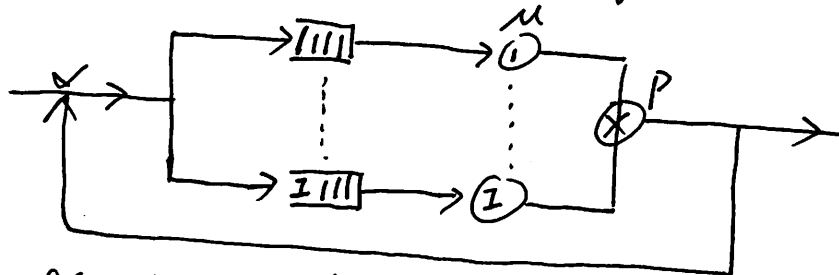
[29/Apr/2014:13:44:36 -0700] - p5880c-315883 - math147.ucsd.edu

05/13/2014 Tuesday

LCP: S-matrix vs nacc + snft for existence for all $x_0 \in \mathbb{R}^d$
 P-matrix vs uniqueness"

Ref: Cottle, Pang, Stone, Academic Press 1992

Fluid model for a single class open queueing network



$$R = (I - P') \text{diag}(u)$$

A fluid model soln associated with $x_0 \in \mathbb{C}_+^d$
 is a pair $(z, y) \in \mathbb{C}_+^{2,d}$; $z(t) = x(t) + R y(t)$,
 where y is non-decreasing, starts from 0 & $\int_0^\infty z_i(s) dy_i(s) < \infty$
 showed last time that:

$$(\bar{Q}^n, \bar{I}^n) \Rightarrow (\bar{Q}, \bar{I}) \text{ as } n \rightarrow \infty$$

where $(\bar{Q}, \bar{I})$ is the unique soln of sp. with

$$x(t) = z_0 + \lambda t - (I - P') \lambda t, \quad t \geq 0.$$

$$\& R = (I - P') \text{diag}(u) \text{ provided } \bar{Q}^n(0) \Rightarrow z_0$$

Dai's Thm (see Branson) 1995

The fluid model is stable if $\exists N$ s.t. $\bar{Q}(t) = 0, \forall t \geq N / |\bar{Q}(0)|$
 for all fluid model solns $\bar{Q}$.

Assume mild conditions: interarrival times are unbounded
 & for each $i=1, \dots, I$,

spread out condition

$\exists h_i$ s.t. $P(u_i(1) + \dots + u_i(h_i) \in [c, d]) \geq \int_c^d f_i(s) ds$
 for some $f_i \geq 0$ satisfying $\int_0^\infty f_i(s) ds > 0$ for all $0 \leq c < d < \infty$
 Suppose fluid model is stable, then the original single class queueing network has a Markovian state descriptor that is positive Harris recurrent.

Now, back to the fluid model:

λ_i = average nominal rate of flow into buffer i

$$\lambda = \alpha + P' \lambda \quad \& \quad \lambda_i = \alpha_i + \sum_j P_{ji} \lambda_j \quad \text{--- traffic eqn}$$

$$\Downarrow$$

$$\lambda = (I - P')^{-1} \alpha$$

For stability, need $\lambda_i < \mu_i, \forall i$

Assume $\lambda < \mu$, want to show fluid model is stable

let $f(t) = (\mu^{-1})' (I - P')^{-1} \bar{Q}(t)$. for a given fluid model

Note that $(I - P')^{-1} \bar{Q}(t) = (I + P' + P'^2 + \dots) \bar{Q}(t)$, ^{solve $\bar{Q}$}

which is interpreted as the average expected amount of work for system associated with jobs recently currently there

$$\bar{Q}(t) = \bar{Q}(\infty) + \alpha t - (I - P') \mu t + (I - P') \text{diag}(\mu) \bar{I}(t)$$

$\bar{I}_i$ can only increase when $\bar{Q}_i = 0, \forall i$

Define fluid departure function:

$$\bar{D}(t) = \text{diag}(\mu) (t - \bar{I}(t)) \quad \text{--- departure process}$$

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Dropbox - 185-midterm1-2013.pdf

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Then $\bar{Q}(t) = \bar{Q}(0) + \alpha t - \bar{D}(t) + \underbrace{P' \bar{D}(t)}_{\text{corresponds to feedback.}}$

$\bar{Q}, \bar{D}, \bar{I}$ are all unof. lip. fcts. hence abs. fcts.

Consider a time $t > 0$ where $\bar{Q}, \bar{I}, \bar{D}$ are differentiable

$$\dot{\bar{Q}}(t) = \frac{d}{dt} \bar{Q}(t) = \alpha - \dot{\bar{D}}(t) + P' \dot{\bar{D}}(t)$$

Let $G = \{i: \bar{Q}_i(t) > 0\}$

$$0 = \dot{\bar{Q}}_G(t) = \alpha_G - \dot{\bar{D}}_G(t) + P'_{G^c G} \dot{\bar{D}}_G(t) + P'_{G G} \dot{\bar{D}}_G(t)$$

Now, $\dot{\bar{I}}_G(t) = 0 \Rightarrow \dot{\bar{D}}_G(t) = \mu_G$

$$\Rightarrow 0 = \alpha_G - (I - P')_{G^c G} \dot{\bar{D}}_G(t) + P'_{G G} \mu_G$$

$$\Rightarrow \dot{\bar{D}}_G(t) = (I - P')_{G^c G}^{-1} (\alpha_G + P'_{G G} \mu_G)$$

$$\geq (I - P')_{G^c G}^{-1} (\alpha_G + P'_{G G} \lambda_G) \text{ since } \lambda < \mu.$$

$$\begin{aligned} \lambda_{G^c} = \alpha_{G^c} &\quad \downarrow \\ + P'_{G^c G} \mu_G &= (I - P')_{G^c G}^{-1} (\alpha_{G^c} + \lambda_{G^c} - \alpha_{G^c} - P'_{G^c G} \mu_G) \\ + P'_{G^c G} \mu_G &= \lambda_{G^c} \end{aligned}$$

$$\dot{\bar{D}}_G(t) = \mu_G \Rightarrow \dot{\bar{D}}(t) \geq \lambda \text{ \& " > " for at least one component unless } \bar{Q}(t) \equiv 0.$$

Now, consider $\dot{f}(t) = (U^{-1})' (I - P')' \dot{\bar{Q}}(t)$

$$= (U^{-1})' (I - P')^{-1} (\alpha - (I - P') \dot{\bar{D}}(t))$$

$$= (U^{-1})' (\lambda - \dot{\bar{D}}(t))$$

$$\leq -\varepsilon, \quad \varepsilon = \min_{i=1}^I (\mu_i^{-1} (\mu_i - \lambda_i))$$

whenever $\bar{Q}(t) \neq 0$.

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multiple-sample-numerical.pdf

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It follows that

$$\bar{Q}(t) = 0 \text{ for all } t \geq \frac{\mu^{-1}(1-p')\bar{Q}(0)}{\epsilon}$$

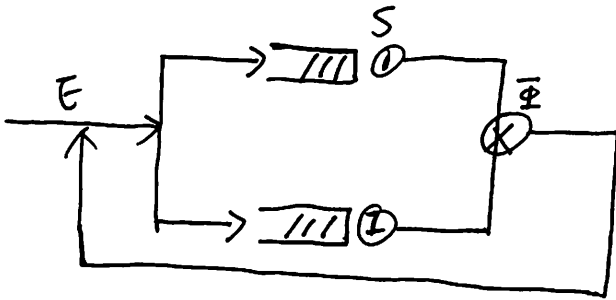
$$\Rightarrow \exists N. \text{ s.t. } \bar{Q}(t) = 0, \quad \forall t \geq N|\bar{Q}(0)|$$

Thus fluid model is stable.

Diffusion Approximation for $\lambda = \mu$.

Consider a sequence of single class queueing models where initial condition changes with n .

$$Q_i^n(t) = Q_i^n(0) + E_i(t) - S_i(T_i^n(t)) + \sum_{j=1}^I \Phi_i^j(S_j(T_j^n(t)))$$



Diffusion scaling:

$$\hat{Q}_i^n(t) \triangleq \frac{Q_i^n(n^2 t)}{n} = \hat{Q}_i^n(0) + \hat{E}_i^n(t) \dots$$

$$\hat{E}_i^n(t) = \frac{E_i(n^2 t) - \alpha_i n^2 t}{n}$$

$$\hat{S}_i^n(t) = \frac{S_i(n^2 t) - \mu_i n^2 t}{n}$$

$$\hat{\Phi}_i^j(x) = \frac{\Phi_i^j(n^2 x) - \rho_{ij} n^2 x}{n}$$

$$\hat{T}_i^n(t) = \frac{T_i^n(n^2 t)}{n^2}$$

Therefore:

$$\begin{aligned} \hat{Q}_i^n(t) \triangleq \frac{Q_i^n(n^2 t)}{n} &= \hat{Q}_i^n(0) + \hat{E}_i^n(t) - \hat{S}_i^n(\hat{T}_i^n(t)) + \sum_{j=1}^I \hat{\Phi}_i^j(\hat{S}_j^n(\hat{T}_j^n(t))) \\ &\quad + n(\alpha_i t - \mu_i \hat{T}_i^n(t)) + \sum_{j=1}^I \rho_{ij} \hat{S}_j^n(\hat{T}_j^n(t)) \end{aligned}$$

Assume $\hat{Q}^n(\omega) \Rightarrow \bar{Q}(\omega)$ as $n \rightarrow \infty$.

$\bar{T}^n(\cdot) \Rightarrow \bar{T}(\cdot)$ as $n \rightarrow \infty$ where $\bar{T}(t) = 1t$

Can show $(\bar{Q}^n, \bar{T}^n, \bar{I}^n) \Rightarrow (\bar{Q}, \bar{T}, \bar{I})$

where $\bar{Q}_i(t) = 0 + \alpha_i t - \mu_i \bar{T}_i(t) + \sum_{j \neq i} p_{ji} \mu_j \bar{T}_j(t)$

Can show $\bar{Q}_i(t) = 0 \forall t$ for $\lambda = \mu$.

$\Rightarrow \alpha = (I - P') \text{diag}(\mu) \bar{T}(t) \Rightarrow \bar{T}(t) = \text{diag}(\mu^{-1}) (I - P')^{-1} \alpha t$

From standard FCLT:

$(\hat{E}^n, \hat{S}^n, \hat{\Phi}^{n,1}(\cdot), \hat{\Phi}^{n,I}(\cdot), \bar{T}^n(\cdot), \hat{Q}^n(\omega)) = 1t$

$\Rightarrow (\tilde{E}(\cdot), \tilde{S}(\cdot), \tilde{\Phi}'(\cdot), \dots, \tilde{\Phi}^I(\cdot), 1(\cdot), \tilde{Q}(\omega))$

BM cov. matrix
 $\text{diag}(\lambda_i^2 \sigma_{a,i}^2)$

BM cov. matrix
 $\text{diag}(\mu_i^2 \sigma_{s,i}^2)$

BM. jth one has cov. matrix with
k-l entry $\begin{cases} p_{jk} (1 - p_{jk}), & \text{if } k=l \\ -p_{jk} p_{jl}, & \text{if } k \neq l \end{cases}$

Moreover

$$\begin{aligned} & n(\alpha_i t - \mu_i \bar{T}_i^n(t) + \sum_{j \neq i} p_{ji} \mu_j \bar{T}_j^n(t)) \\ & \stackrel{\textcircled{1}}{=} n\alpha_i t - \mu_i n t + \mu_i \hat{I}_i^n(t) + \sum_j p_{ji} \mu_j n t - \sum_j p_{ji} \mu_j \hat{I}_j^n(t) \\ & = n t (\alpha_i - ((I - P')\mu)_i) + ((I - P') \text{diag}(\mu) \hat{I}^n(t))_i \\ & \quad \text{" } \circ \text{ since } \lambda = \mu. \end{aligned}$$

$$\begin{aligned} \textcircled{1} \text{ holds since } \hat{I}_i^n(t) &= \frac{n^2 t - T_i^n(n^2 t)}{n} = n t - n \bar{T}_i^n(t) \\ \Rightarrow n \bar{T}_i^n(t) &= n t - \hat{I}_i^n(t) \end{aligned}$$

So. $\hat{Q}^n(t) = \hat{X}^n(t) + R \hat{I}^n(t)$, $R = (I - P')$ diag(μ)
 where $\hat{X}^n \Rightarrow \tilde{X}$.

$$\tilde{X} = \tilde{Q}(\omega) + \tilde{E}(t) - \tilde{S}(t) + \sum_{j=1}^I \tilde{\Phi}^j(\mu_j, t) + P' \tilde{S}(t)$$

$\hat{I}^n$ is non-decreasing

$$\hat{I}^n(0) = 0, \quad \int_{(0, \infty)} \hat{Q}_i^n(s) d\hat{I}_i^n(s) = 0. \quad \forall i$$

$(\hat{Q}^n, \hat{I}^n)$ is soln of s.p. for $\hat{X}^n \Rightarrow (\hat{Q}^n, \hat{I}^n) = (\varphi, \psi)(\hat{X}^n)$

Since (φ, ψ) is continuous, by its mapping thm.

$$(\hat{Q}^n, \hat{I}^n) = (\varphi, \psi)(\hat{X}^n) \Rightarrow (\tilde{Q}, \tilde{I}) = (\varphi, \psi)(\tilde{X})$$

In particular, $\hat{Q}^n \Rightarrow \tilde{Q} \leftarrow \text{SRBM} \leftarrow \begin{cases} \text{zero drift} \\ \text{reflection matrix } R \end{cases}$