

4/24

- NEXT THURSDAY 1PM IN AP+M 6402 MICHAEL KINNALLY

- NO CLASS NEXT TUES

One dimensional Skorokhod problem

$$\text{Let } \mathcal{D}_+ = \{ x : [0, \infty) \rightarrow \mathbb{R}, x \text{ r.c.l.l., } x(0) \geq 0 \}$$

Def Let $x \in \mathcal{D}_+$. A pair $(z, y) \in \mathcal{D}_+ \times \mathcal{D}_+$ is a solution of the (one-dim) Skorokhod problem for x if the following hold:

i) $z(t) = x(t) + y(t), t \geq 0$

ii) $z(t) \geq 0, t \geq 0$ ↖ regulator for x

iii) y satisfies

a) $y(0) = 0$

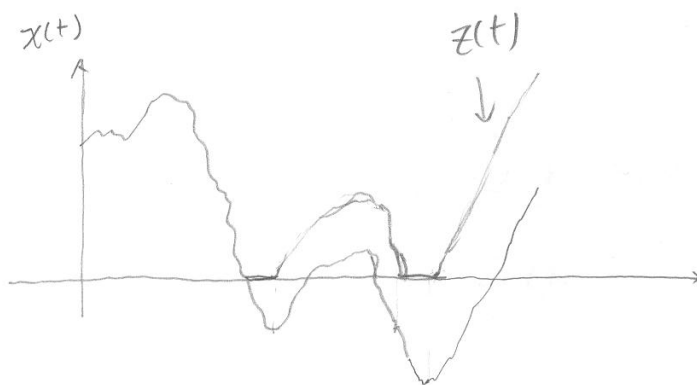
b) y is non-decreasing

c) $\int_{(0, \infty)} z(t) dy(t) = 0$

(y can increase only when z is zero)

$\Leftrightarrow$

$$\int_{[0, \infty)} z(t) dy(t) = 0 \quad (\Delta(y(0)) = y(0) - y(0^-) = 0)$$



Lemma

Suppose $(z, y) \in D_+ \times D_+$

s.t. $z(t) \geq 0$,

$y(0) = 0$, y is non-decreasing

then $\int_{(0, \infty)} z(t) dy(t) = 0$ (1)

if and only if

(2) * (a) $z(t) = 0$ for each $t > 0$ s.t. $\Delta y(t) \triangleq y(t) - y(t^-) > 0$

* (b) $y^c(t) = y^c(s)$ whenever $z(u) > 0$ for all $u \in [s, t]$

where $y = y^c + y^d$, $y^d(t) = \sum_{0 \leq s \leq t} \Delta y(s)$

~~Proof~~: Suppose $f: [0, \infty) \rightarrow \mathbb{R}$ is right continuous & locally of bounded variation.

Then (*) $(f(t))^2 - (f(0))^2 = 2 \int_{(0,t]} f(s) df(s) - \int_{0 \leq s \leq t} (\Delta f(s))^2$

(Exercise: Suppose $f(t) = \sum_{0 \leq s \leq t} \Delta f(s)$, check (*))

Proof

Suppose (2) holds.

↓ Riemann-Stieltjes

$$\int_{(0,t]} Z(s) dy(s) = \int_{(0,t]} Z(s) dy^c(s) + \int_{(0,t]} Z(s) dy^d(s)$$

$$= \lim_{n \rightarrow \infty} \sum_{i=1}^n \inf_{s \in [t_i^n, t_{i+1}^n]} Z(s) (y^c(t_{i+1}^n) - y^c(t_i^n)) + \sum_{0 \leq s \leq t} Z(s) \Delta y(s)$$

$\xrightarrow{\text{by (2) (a)}} 0$ $\xrightarrow{\text{by (2) (a)}} 0$

(If $\inf Z(s) > 0$, then $Z(u) > 0$ for $u \in [t_i^n, t_{i+1}^n)$
 then by (2) (b), $y^c(t) = \text{const}$ on $[t_i^n, t_{i+1}^n]$)

So (1) holds.

Suppose (1) holds.

$$0 = \int_{(0,t]} z(s) dy(s) = \underbrace{\int_{(0,t]} z(s) dy^c(s)}_{i) \geq 0} + \underbrace{\int_{(0,t)} z(s) dy^d(s)}_{ii) \geq 0} \quad \forall t.$$

$$i) \Rightarrow 0 = \int_{(0,t]} z(s) dy^d(s) = \sum_{0 < s \leq t} z(s) \Delta y(s)$$

$$\Rightarrow z(s) \cdot \Delta y(s) = 0 \quad \forall s > 0 \Rightarrow (2)(a) \text{ holds.}$$

$$ii) \int_{(0,t]} z(s) dy^c(s) = 0 \quad \forall t \quad [\text{proof by contradiction}]$$

Suppose (2) (b) does not hold, i.e.,

$$\exists 0 < s \leq t \text{ st. } z(u) > 0 \text{ on } [s, t], \quad y^c(t) > y^c(s)$$

$$\text{Then } \int_{[s,t]} z(u) dy^c(u) > 0 \quad \rightarrow \leftarrow$$

So (2) (b) must hold.

Theorem

Let $x \in \mathbb{D}_+$, then there exists a unique pair $(z, y) \in \mathbb{D}_+ \times \mathbb{D}_+$ that solves Skorokhod problem for x .

$$\text{In fact, } y(t) = \sup_{0 \leq s \leq t} x^-(s)$$

$$\text{where } x^-(s) = \max(-x(s), 0)$$

Proof: Suppose $(z, y) + (\tilde{z}, \tilde{y})$ are solutions of SP for x .

$$\text{Let } f = z - \tilde{z} = y - \tilde{y} \quad (z(t) = x(t) + y(t))$$

$\Rightarrow f$ is r.c. + locally of bounded variation

$$0 \leq f^2(t) + \sum_{0 \leq s \leq t} (\Delta f(s))^2 = 2 \int_{(0, t]} f(s) df(s)$$

$$= 2 \int_{(0, t]} (z - \tilde{z})(s) d(y - \tilde{y})(s)$$

$$= 2 \int_{(0, t]} z(s) dy(s) + 2 \int_{(0, t]} \tilde{z}(s) d\tilde{y}(s)$$

$$- 2 \int_{(0, t]} z(s) d\tilde{y}(s) - 2 \int_{(0, t]} \tilde{z}(s) dy(s)$$

$\geq 0 \qquad \qquad \qquad \geq 0$

$$\Rightarrow f(t) = 0 \quad \forall t.$$