

05/01/2014 Thursday

One-dimensional Skorohod Problem (sp).

Given $x \in \mathbb{D}_+$, a solution to the sp for x is a pair $(y, z) \in \mathbb{D}_+ \times \mathbb{D}_+$ satisfying

(i) $z = x + y$

(ii) $z(t) \geq 0$ for all $t \geq 0$

(iii) $y(0) = 0$. y is non-decreasing & $\int_{(0, \infty)} z(s) dy(s) = 0$.

Thm: Given $x \in \mathbb{D}_+$, there is a unique solution (y, z) of sp. for x .

In particular, $y(t) = \sup_{0 \leq s \leq t} \max_{s \leq t} x^-(s) \dots (1)$

Proof of existence: (last time we've proved uniqueness)
Suppose y is given by (1). & let $x + y = z$

(i) is trivial.

For (ii). $z(t) = x(t) + y(t) \geq x(t) + x^-(t) \geq x(t) - x(t) = 0$

For (iii) $y(0) = x^-(0) = \max(-x(0), 0) = 0$. Since $x(0) \geq 0$.
 y is clearly non-decreasing.

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To prove $\int_{(0, \infty)} Z(s) dy(s) = 0$.

or equivalently: (a) $Z(t) = 0$ if $\Delta y(t) > 0$ for some $t > 0$

(b) whenever $Z(u) > 0$ for all $u \in [s, t]$,

then $y^c(s) = y^c(t)$, for $0 \leq s < t < \infty$
(i.e. continuous part is flat)

For (a): consider $t > 0$ s.t. $\Delta y(t) > 0$

then $y(t) > y(t-) = \sup_{0 \leq s < t} x^-(s)$

$\Rightarrow x^-(t) > \sup_{0 \leq s < t} x^-(s)$ & $\underbrace{x^-(t)}_{\Rightarrow x^-(t) = -x(t)} > 0$

$\Rightarrow y(t) = x^-(t) = -x(t)$

$\Rightarrow Z(t) = x(t) - x(t) = 0$, thus (a) is checked.

For (b). for a proof by contradiction, suppose

$\exists 0 \leq s < t < \infty$, s.t. $Z(u) > 0$ for $u \in [s, t]$ & $y^c(t) > y^c(s)$

By (a), $\Delta y(u) = 0$ for all $u \in [s, t]$.

So y is cts on $[s, t]$

let $u^* = \inf \{u \geq s : y(u) > y(s)\}$.

Then $y(u) = y(s)$ for $u \in [s, u^*]$

& $u^* < t$ (if $u^* = t$, then y is flat on $[s, t]$,

& which contradicts the fact $y^c(t) > y^c(s)$)

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Also. $y(u) > y(u^*)$ for all $u \in (u^*, t]$
 $\parallel$
 $\sup(y(u^*), -x(v) : u^* \leq v \leq u)$

Then for any $\varepsilon > 0$, we can find $u^\varepsilon \in (u^*, u^* + \varepsilon]$
 s.t. $y(u^\varepsilon) = -x(u^\varepsilon)$.

Let $\varepsilon \downarrow 0$. $y(u^*) = -x(u^*)$. then $z(u^*) = x(u^*) + y(u^*)$

This is a contradiction as desired.

$$= x(u^*) - x(u^*) = 0 \quad \downarrow$$

End of proof.

Notation: $\varphi(x) = z$, $\psi(x) = y$.

Lemma: For $x, \tilde{x} \in ID_+$, $t \geq 0$.

$$\|\varphi(x) - \varphi(\tilde{x})\| \leq \|x - \tilde{x}\|$$

$$\|\psi(x) - \psi(\tilde{x})\| \leq 2\|x - \tilde{x}\|.$$

Multidimensional Skorokhod problem

Def. 3.1: $x \in ID_+^d$, $\varphi(z, y) \in ID_+^d \times ID_+^d$.

(i). $z(t) = x(t) + R y(t)$, $t \geq 0$ where R is a given $d \times d$ matrix

(ii) $z(t) \geq 0$, $t \geq 0$. (i.e. $z_i(t) \geq 0$, $\forall i$, $\forall t \geq 0$)

(iii) y is such that for $i=1, \dots, d$, $y_i(0) = 0$.

$$\cancel{y_i \geq 0}. \quad y_i \text{ non-decreasing} \\ \& \int_{0, \infty} z_i(s) dy_i(s) = 0.$$

then the solution (z, y) is called an R -regulation for x .

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Def 4.1: A $d \times d$ matrix $R \in \mathbb{R}^{d \times d}$ is called a generalized M -matrix if

(i) All diagonal entries are > 0 .

(ii) if Q is such that $(I - Q) \text{diag}(R) = R$, then the spectral radius of $|Q|$ is less than 1. strictly

Remark: R specify the direction that $y(x)$ is ~~pushed in at~~ ~~the boundary~~ ~~both~~ (intuitively).

Theorem 4.2: Suppose $R_{d \times d}$ is a generalized M -matrix. Then for each $x \in \mathbb{R}_+^d$, an R -regulation of x exists and is unique. Moreover, ϕ & ψ associated with R are well-defined.

Proof of Thm.

Reductions:

o Assume $\text{diag}(R) = I$; Actually, if (z, y) is R -regulation of x

o Assume $\|Q\| = \max_i \sum_{j=1}^d |q_{ij}| < 1$ then $(z, \text{diag}(R)y)$ is $R \text{diag}(R)^T$ -regulation of x

(Veinott (1969), Lemma 3).

Otherwise, $\tilde{Q} \doteq BQ B^{-1}$, where $B = \text{diag}(b_1, \dots, b_d)$, $b_i > 0$. then $\|\tilde{Q}\| < 1$

Suppose (Bz, By) is an $(I - \tilde{Q})$ -regulation of Bx

$$Bz = Bx + (I - \tilde{Q})By = Bx + By - BQ B^{-1}By = Bx + B(I - Q)y$$

$\Rightarrow z = x + (I - Q)y$, (z, y) is $(I - Q)$ -regulation of x .

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$$\mathbb{D}_0^d = \{y \in \mathbb{D}^d \mid y \text{ non-decreasing, } y_0 = 0\}$$

Define: $\pi: \mathbb{D}_0^d \rightarrow \mathbb{D}_0^d$ by $(\pi(y)(t))_i = \sup_{0 \leq s \leq t} (x_i(s) - (Qy)_i(s))^-$

Claim: If (z, y) satisfies $y \in \mathbb{D}_0^d$, $z = \pi(y)$.

$z = x + (I - Q)y$, then (z, y) is an $(I - Q)$ -regulation

$z(t) = x + y - Qy(t) = y + (x - Qy)$ for x .

$= \pi(y)(t) + (x - Qy)(t) \geq 0$.

If $x(t)Qy(t) < 0$, then $\pi(y)(t) \geq (x(t) - Qy(t))^-$

Now, the first two conditions are checked for Def 3.1.

For the third condition in Def 3.1

we want to show, $\forall i = 1, \dots, d$, $\int_{(0, \infty)} z_i(t) dy_i(t) = 0$,
for which we can treat as a one-dimensional Skorohod
problem.

At last, to justify $y = \pi(y)$.

It suffices to show π is a contraction mapping
w.r.t. $\|\cdot\|_t$, where $\forall x$, $\|x\|_t = \sup_{0 \leq s \leq t} |x(s)|$.

In fact, $|\pi_i(y)(t) - \pi_i(y')(t)|$
 $= \left| \sup_{0 \leq s \leq t} (x_i(s) - (Qy)_i(s))^- - \sup_{0 \leq s \leq t} (x_i(s) - (Qy')_i(s))^- \right|$
 $\leq \sup_{0 \leq s \leq t} |(Qy(s))_i - (Qy'(s))_i|$, since $|a^- - b^-| \leq |a - b|$

$$\forall t \in [0, T)$$

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$$|\pi(y)(t) - \pi(y')(t)| \leq \max_i \sup_{0 \leq s \leq t} |(Qy(s))_i - (Qy'(s))_i|$$

$$= \sup_{0 \leq s \leq t} \max_i |(Qy(s))_i - (Qy'(s))_i|$$

$$= \sup_{0 \leq s \leq t} \| (Qy(s)) - (Qy'(s)) \|_{\infty} ; \quad \|Q\| < 1.$$

$$\leq \|Q\| \sup_{0 \leq s \leq t} |y(s) - y'(s)|$$

$$\|x\|_{\infty} = \max_i |x_i|.$$

$$\Rightarrow \|\pi(y) - \pi(y')\|_T \leq \|Q\| \cdot \|y - y'\|_T$$

then by the fact that $(\mathbb{D}_0^d, \|\cdot\|_T)$ is a complete metric space.
and using fixed-point theorem. the uniqueness is proved.

Remark:

and existence of $y = \pi(y)$

(1). ϕ & ψ are continuous w.r.t. J_1 -topology.

and also uniformly Lipschitz as *when time is confined to a bounded interval.*

(2). (Z, γ) , R -reg w.r.t. α . then

$(Z \circ \gamma, \gamma \circ \gamma)$ is R -reg w.r.t. $\alpha \circ \gamma$.

where $\gamma: (0, \infty) \rightarrow (0, \infty)$ is a time change.