

Given $d \times d$ matrix R , $x \in \mathbb{R}^d$.

A solution of the LCP for x is a pair $(z, y) \in \mathbb{R}^{2d}$ such that

(i) $z = x + Ry$

(ii) $z \geq 0$ $y_i = 0 \text{ if } z_i > 0$

(iii) $y \in S$ ($y_i \geq 0 \forall i$), $z \cdot y = 0$ (i.e., $z_i y_i = 0 \forall i$)

LCP has a solution for each $x \in \mathbb{R}^d$ iff

R is ~~completely~~ ^{soln} ~~an~~ S -matrix.

LCP has a unique ^{soln} for each $x \in \mathbb{R}^d$ iff

R is a P -matrix (i.e., all principal minors are positive)

Even if R is a P -matrix, SP need not have a unique solution (given $x \in \mathbb{R}^d$)

→ see example from previous lecture: $R = \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}$

→ if R not P -matrix, easy to find examples for non-unique SP solutions: $R = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

Theorem:

For each $x \in \mathbb{R}^d$, there is a solution of the SP for x when R is completely S .

Exercise: prove above theorem.

①

Oscillation Inequality:

Assume R is completely S and $\delta \geq 0$.

$$0 \leq t_1 < t_2 < \infty,$$

$$x, z, y \in D([t_1, t_2], \mathbb{R}^d)$$

$$= \{w : [t_1, t_2] \rightarrow \mathbb{R}^d, w \text{ is s.c.l.t.}\}$$

are such that

$$(i) \quad z(t) = x(t) + R y(t), \quad t \in [t_1, t_2]$$

$$(ii) \quad z(t) \in S \quad \forall t \in [t_1, t_2]$$

(iii) for $i=1, \dots, d$:

$$(a) \quad y_i(t_1) \geq 0,$$

(b) y_i is non-decreasing,

$$(c) \quad \int_{[t_1, t_2]} 1_{(0, \infty)}(z_i(s)) dy_i(s) = 0$$

$\rightarrow y_i$ can only increase when $z_i \leq 0$

Then $\exists C > 0$ that depends only on R

$$\text{such that } \text{Osc}(y, [t_1, t_2]) \leq C(\text{Osc}(x, [t_1, t_2]) + \delta)$$

$$\text{Osc}(z, [t_1, t_2]) \leq C(\text{Osc}(x, [t_1, t_2]) + \delta).$$

$$\text{Osc}(f, [t_1, t_2]) = \sup \{ |f(t) - f(s)| : t_1 \leq s < t \leq t_2 \}$$

For proof see SP notes.

(2)

Theorem (Taylor - Williams, '93):

Suppose R is completely S .

There is an SRBM associated with

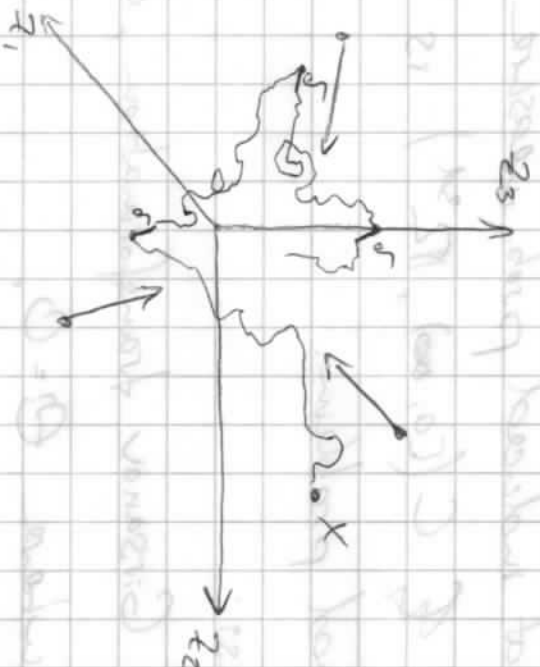
$(S, \mathcal{E}, \Gamma, R)$ and it is unique in law.

$(\Rightarrow P_x(Z(\cdot) \in A) = \tilde{P}_x(\tilde{Z}(\cdot) \in A))$
 $\forall A$ measurable sets $A, x \in S$

Furthermore, it defines a strong Markov process.

Proof sketch:

• Existence:



Fix $\delta, x \in S$. Let $X_h \in (\mathcal{E}, \Gamma)$ Brownian motion starting from x .

Define a process Z^δ that uses the increments of X to move when Z^δ is in the interior of S , whenever Z^δ reaches a boundary face, jump up to distance δ into S in a direction given by a convex combination of directions of reflection that you are allowed to use there. (3)

$$Z^0 = X + \mathbb{R} Y^0$$

Z^0 satisfies conditions of oscillation inequality.

Use oscillation inequality to show that

for $\partial_n \downarrow 0$ as $n \rightarrow \infty$,

$\{(Z^0_n, Y^0_n, X)\}_{n=1}^{\infty}$ is C -tight.

Any limit point (Z^*, Y^*, X^*) is an

SREBM starting from X .

Look at induced prob. measure on canonical path space $\mathbb{C}([0, \infty), \mathbb{R}^{3d})$ is $\mathbb{P}_x$ for standard problem.

• Uniqueness:

→ Using Girsanov transformation, can reduce to case where $\partial = 0$.

→ Use an inductive argument to prove

dimension by dimension

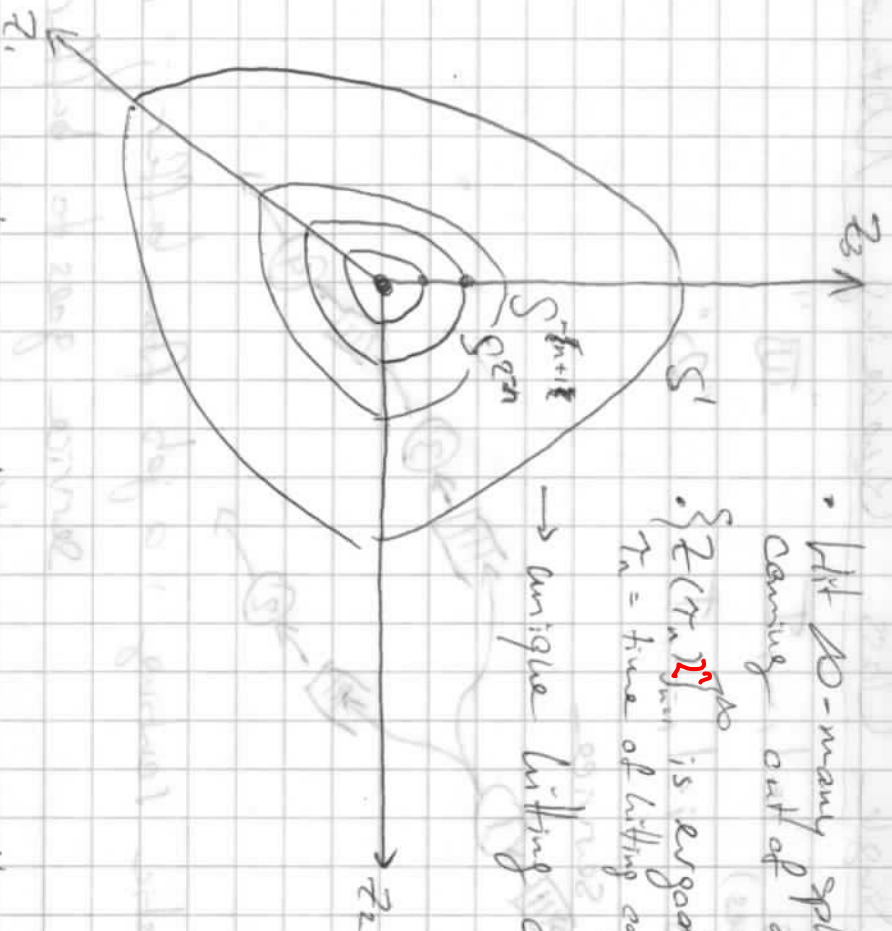
→ Can reduce using an induction hypothesis to prove uniqueness starting from the origin.

④

• Let 10-many spherical caps coming out of origin

• $\{Z(t_n)\}_{n=1}^{\infty}$ is ergodic
 $t_n = \text{time of hitting cap of radius } 2^{-n}$

→ unique lifting distribution



Do a proof by contradiction → if there were

the SRBMs starting from the origin, have same lifting distributions on S'

→ contradiction

→ works by scaling property of BM:

$$\frac{B(c \cdot)}{\sqrt{c}} = B(\cdot)$$

(5)

(6)

Open Single Class Queueing Network:

Buffers
(classes)
 $i = 1, \dots, I$

"III"

Servers $j = 1, \dots, J$

"O"

FIFO service



Probabilistic routing: a job from buffer i after

service goes to buffer j

with probability P_{ij} spectral radius < 1

Arrivals: for each i , there is an exogenous

arrival process

$\{E, (t)\}$: # exogenous arrivals to buffer i up

to time t .

Renewal process $= \sup \{m: \sum_{i=1}^m u_i(t) \leq t\}$, where

$\{u_i(t)\}_{i=1}^\infty$ are i.i.d. interarrival times with mean $1/\mu_i$, variance $\sigma_{a,i}^2$.

Service: ~~the~~ i.i.d. sequence of service times for

class i : jobs, $\{v_i(t)\}_{i=1}^\infty$ with finite

mean $1/\mu_i$, variance $\sigma_{s,i}^2$.

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$S_i(t) = \#$ jobs served by server i up to t
 if server never idles
 $= \sup_{t \leq u} \{u - \sum_{l=1}^m v_l(t) \mid t + \}$

Routing: For $j=1, \dots, T$, let
 $\{\varphi^j(l), l=1, 2, \dots\}$ be i.i.d. sequence
 of random vectors s.t. $\varphi^j(l)$ has all
 entries ~~is~~ zero except possibly one
 which takes the value 1.
 $(\varphi^j(l))_{l \in \mathbb{N}} \in \{e_1, e_2, \dots, e_T, e_0\}$ where $e_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{R}^I$ with
 $\rightarrow$ let $\Phi^j(u) = \sum_{l=1}^u \varphi^j(l)$ $e_0 = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$

Performance measure:

$$\begin{aligned}
 Q_i(t) &= \# \text{ jobs of class } i \text{ at time } t \\
 &= Q_i(0) + E_i(t) - S_i(\overleftarrow{T_i(t)}) \\
 &\quad + \sum_{j=1}^T \Phi^j(S_i(T_j(t))) \\
 \underline{T}_i(t) &= \text{cumulative idle time for server } i \\
 &= t - T_i(t)
 \end{aligned}$$

$\overleftarrow{T_i(t)} = \int_0^t \mathbb{1}_{\{Q_i(s) > 0\}} ds$, busy time for server i

Assume $(E_1, \dots, E_I, S_1, \dots, S_I, \Phi^1, \dots, \Phi^T)$ are
 mutually independent.

⑦

⑧

Fluid Scaling:

Consider a sequence of such systems where $Q(c)$ changes along the sequence, denote it by $Q^n(c)$. Then

$$Q_i^n(t) = Q_i^n(c) + E_i(t) - S_i(T_i^n(t)) + \sum_{j=1}^I \bar{\Phi}_j(S_j(T_j^n(t)))$$

Fluid ~~scaling~~ scaling:

$$\bar{Q}^n(t) = \frac{Q^n(ut)}{n} \text{ etc. , then}$$

$$\bar{Q}_i^n(t) = \bar{Q}_i^n(c) + \bar{E}_i^n(t) - \bar{S}_i^n(T_i^n(t)) + \sum_{j=1}^I \bar{\Phi}_j^n(S_j^n(T_j^n(t)))$$

Suppose $\bar{Q}^n(c) \Rightarrow q_0$

$$(E^n(c), S^n(c), \bar{\Phi}^{1,n}(c), \dots, \bar{\Phi}^{I,n}(c)) \Rightarrow (\alpha(c), \mu(c), (\bar{\Phi}^T)^{(1,n)}(c), \dots, (\bar{\Phi}^T)^{(I,n)}(c))$$

$\{\bar{T}^n(c)\}_{n=1}^\infty$ are $\mathcal{L}$ -tight because they are

" $\bar{T}^n(c)$ " uniformly Lipschitz continuous

$\rightarrow \{(E^n(\cdot), S^n(\cdot), \bar{\Phi}^1(\cdot), \bar{T}^n(\cdot), Q^n(c))\}_{n=1}^\infty$ is $\mathcal{L}$ -tight

Consider a limit point $\int_{\text{diag}(\mu)} (\bar{E}, \bar{S}, \bar{T}, \bar{Q})$, then

$$\bar{Q}(t) = \bar{Q}(c) + \alpha t - \int_{\text{diag}(\mu)} \bar{T}(t) + \sum_{j=1}^I (P^j)^j \mu_j \bar{T}_j(t)$$

$$\bar{T}(t) = \bar{T} - \bar{T}(t) = \underbrace{P^j \text{diag}(\mu) \bar{T}(t)}_{= P^j \text{diag}(\mu) \bar{T}(t)}$$

$$\Rightarrow \bar{Q}(t) = q_0 + \alpha t - (I - P^T) \text{diag}(\mu) t + (I - P^T) \text{diag}(\mu) \bar{T}(t)$$

$\rightarrow \bar{T}_i$ increases only when $\bar{Q}_i$ is zero

Ⓕ (2.4) solve SP for x , unique since Q is an M -matrix

Hence

$$(\overline{E}^n, \overline{S}^n, \overline{F}^n, \overline{T}^n, \overline{Q}^n, \overline{I}^n)$$

$\Rightarrow (\overline{E}, \overline{S}, \overline{F}, \overline{T}, \overline{Q}, \overline{I})$, where

$\overline{G}, \overline{I}$ are unique solution of SP for X .

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