

May 6th 2014

Jiayi Wen.

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Comment to last time.

- Existence of solution of SP for generalized M-matrix
Harrison Reiman (1981)
- Fix pt theorem, contraction argument - Banach.

How to use the Thm proved last time.

- Semimartingale Reflecting Brownian motion (SRBM)

papers about it on webpage.

- survey (R. Williams)
- Brownian particles on the real line with rank dependent drift (Pat + Pitman)

 ← gap process (RBM)

SRBM

fix dim d (positive integer)

Let $S = \{x \in \mathbb{R}^d : x_i \geq 0, i=1, \dots, d\}$

Let $\theta \in \mathbb{R}^d$, σ is $d \times d$ non-degenerate covariance matrix
(sym, strictly pos. def.)

R , $d \times d$ matrix

An SRBM with data (S, θ, σ, R) is a cts process, adapted d -dim process Z , defined on a filtered space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\})$ together with a family of prob measures $\{P_x, x \in S\}$ defined there, s.t. for each $x \in S$, under P_x we have:

$$Z(t) \in \mathbb{S},$$

$$Z(t) = X(t) + R Y(t) \quad t \geq 0 \quad (P_x - \text{a.s.})$$

where (i) X is a d -dim BM with a constant drift θ and covariance matrix $\mathcal{Q}$

$\{X(t) - \theta t, \mathcal{F}_t, t \geq 0\}$ is a martingale

under P_x , $X(0) = x$, (P_x -a.s.)

$$X(t) = B(t) + \theta t$$

$$B(t) \sim N(0, \mathcal{Q}t)$$

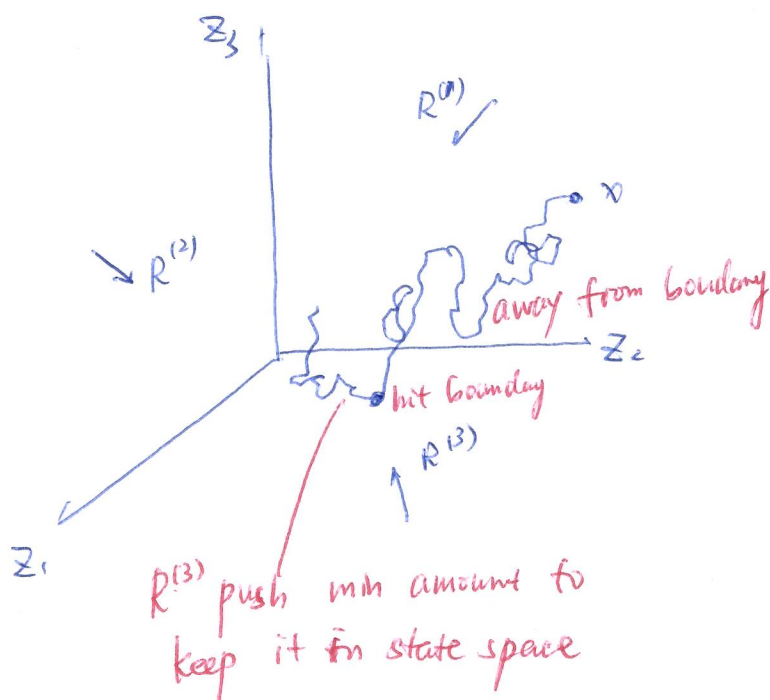
(ii) Y is cts adapted d -dim process s.t.

$$(a) \quad Y(0) = 0$$

(b) non-decreasing (in each component)

$$(c) \quad \int_{(0, \infty)} Z_i(s) dY_i(s) = 0 \quad \forall i=1, \dots, d$$

Behavior of such an SRBM under P_x



$$R = \begin{pmatrix} R^{(1)} & R^{(2)} & R^{(3)} \end{pmatrix}$$

$\uparrow \quad \quad \uparrow \quad \quad \uparrow$
 col 1 col 2 col 3

R : 3×3 matrix, $R^{(i)}$ is a vector on $\{z \in \mathbb{R}^3, z_i = 0\}$

- Away from boundary behavior (like $X(t)$ (BM))
- $RY(t)$ kick in to hold it in the state space

$$\Omega = ([0, \infty), \mathbb{R}^d)$$

$$\mathcal{F} = \sigma\{X(s), 0 \leq s \leq \infty, x \in \Omega\}$$

$$\mathcal{F}_x = \sigma\{X(s), 0 \leq s \leq t, x \in \Omega\}$$

$$P_x, \quad X(t, x) = x(t)$$

Canonical path space.

Then Suppose R is a generalized M -matrix

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(φ, γ) specify the solution of the Skorokhod problem associated with R .

Let $X, \{P_x, x \in S\}$ are defined on a filtered space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\})$
 s.t. under P_x, X is a BM with drift 0, covariance matrix T
 $X(0) = x, P_x$ -a.s. ; $\{X(t) - 0t, \mathcal{F}_t, t \geq 0\}$ is a P_x -martingale.

Let $Y = \gamma(x), Z = \varphi(x)$, Then Z with $\{P_x, x \in S\}$ defines an

SRBM with data $(S, \mathcal{T}, R)$. & it is unique in Law
 (i.e. Z, Z' are SRBM, with $\{P_x, x \in S\}$ & $\{P_{x'}, x' \in S\}$ as prob measures

By uniqueness of the solution of SP, $Z = \varphi(x)$ & $Z' = \varphi(x')$

Since X under $P_x \stackrel{d}{=} X'$ under $P_{x'}$, then Z under $P_x \stackrel{d}{=} Z'$ under $P_{x'}$
 for $x \in S$)

In fact Z under $\{P_x, x \in S\}$ is strong Markov.

↓

If T is a P_x -a.s. finite stopping time
 then $P_x(Z_{T+t} \in A | \mathcal{F}_T) = P_{Z(T)}(\bar{Z}_t \in A) = h^A(Z(T))$
 $h^A(x) = P_x(Z_t \in A)$

$\varphi: \mathbb{D}_+^d \rightarrow \mathbb{D}_+^d, \gamma: \mathbb{D}_+^d \rightarrow \mathbb{D}_+^d$ Given $x \in \mathbb{D}_+^d, Z = \varphi(x), Y = \gamma(x)$
 is solution of the SP for x :

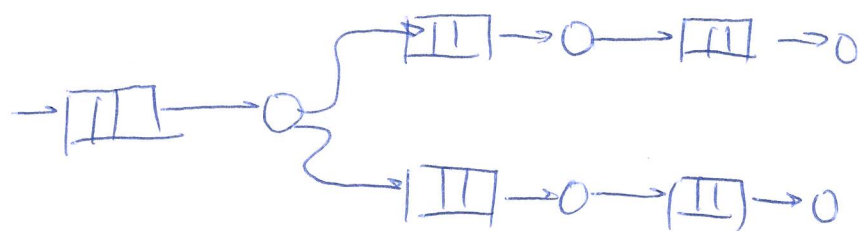
(i) $Z = X + RY$

(ii) $Z(t) \geq 0 \quad \forall t \geq 0$

(iii) $Y_i(t) \geq 0 \quad \forall t \quad Y$ is non-decreasing

$\int_{(0, \infty)} Z_i(s) dY_i(s) = 0 \quad \forall i$

For single class open queueing network



Associated with diffusion approx is an SRBM with $R = (I - P') \text{diag}(\mu)$

μ_i = average service rate at queue

P_{ji} = prob after completing service at server j goes to server i

Open network $\Leftrightarrow P$ has spectral radious < 1

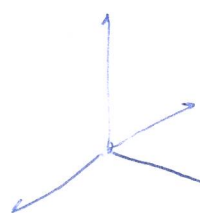
$\Leftrightarrow P$ is a transition matrix for a transient Markov chain

This R is an M -matrix

Multi class queueing network diffusion approx however can lead to R matrix that are not generalized M -matrix

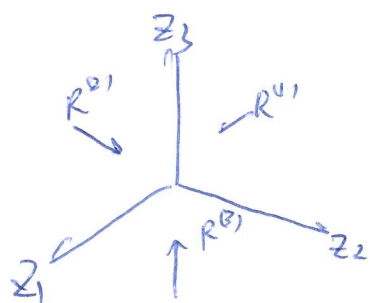
Def A real matrix R is called an S -matrix (S for Stienke) if $\exists v \in \mathbb{R}_+^d$ s.t. $Rv > 0$

R is completely S if every principle submatrix of R is an S -matrix



Principle submatrix are obtained by eliminating all rows + cols of R with a given set of indices

Geometric Interpretation



$R^{(i)}$ is on face where $z_i = 0$

R is completely -S iff at each boundary pt there is a pos. linear combination of the vectors of reflections that one can use there that points into the orthant.

Algebraically

Suppose $z_i \geq 0 \quad \forall i \in A \quad z_i = 0$,
 $\forall i \in A; \exists v_i$ for $i \in A$ st. $v_i \geq 0 \quad \forall i \in A$
 $(\sum_{i \in A} v_i R^{(i)})_j \geq 0$ for $j \in A$

Thm Suppose R is completely -S for each $x \in \mathbb{R}_+^d$, there exists $(y, z) \in \mathbb{D}_+^{2d}$ that solves sp for x ~~problem~~

Note: (y, z) in general is not unique.

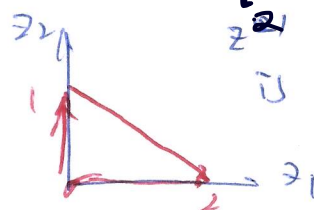
Counter example to uniqueness

(Mandelbaum, van der Heyder)

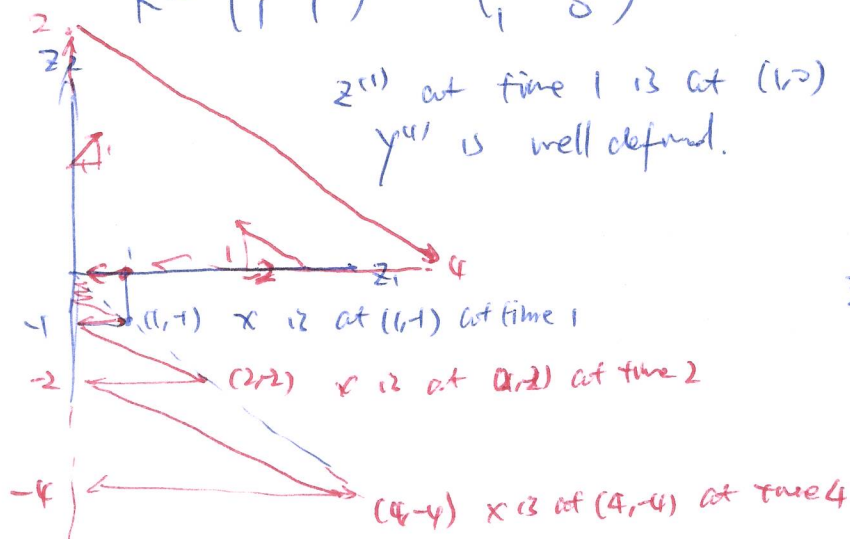
$$R = \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & +2 \\ 1 & 0 \end{pmatrix}$$

non-generalized M-matrix.

at soln
 z_2 at time 1
 z_1 at (0,0)



$z^{(1)}$ at time 1 is at (1,0)
 $y^{(1)}$ is well defined.



zig zag is crucial construction for non-uniqueness.