

van Kampen approximation

(linear noise approx.)
(second order approx)

Fluid scaled system

$$\bar{X}^r(t) = \bar{X}^r(0) + \sum_{k=1}^K (\nu_k' - \nu_k) N_k^r \left(\int_0^t f_k^r(\bar{X}^r(s)) ds \right)$$

$\nu_{ik} = i^{\text{th}}$ component on ν_k

$$f_k^r(x) = \tilde{K}_k \prod_{i=1}^n x_i (x_i - \frac{1}{r}) \dots (x_i - \frac{\nu_{ik}}{r})$$

Assume $\bar{X}^r(0) \Rightarrow \bar{X}(0)$ as $r \rightarrow 0$
 $X_i^r(t) = \#$ molecules of species i in ~~the~~ system at time t

$$\bar{X}^r(t) = \frac{X^r(t)}{r} ; \quad \bar{N}_k^r(\cdot) = \frac{N_k(r, \cdot)}{r}$$

Showed last time that:

$$\bar{X}^r(\cdot) \Rightarrow \bar{X}(\cdot) \quad \text{where} \quad \bar{X}(t) = \bar{X}(0) + \sum_{k=1}^K (\nu_k' - \nu_k) \int_0^t f_k(\bar{X}(s)) ds$$

$$f_k(x) = \tilde{K}_k \prod_{i=1}^n x_i^{\nu_{ik}}$$

(Functional law of large numbers)

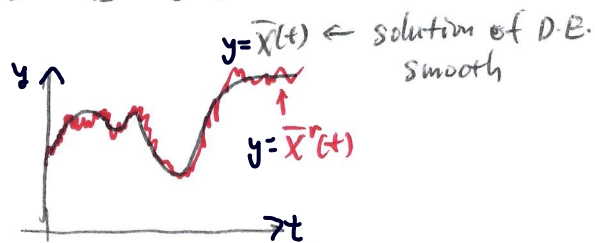
Functional CLT (studying fluctuations)

Kurtz, Stochastic Process + their Applications 6 (1978) 223-240

Let $\hat{V}^r(t) = \sqrt{r} (\bar{X}^r(t) - \bar{X}(t))$

$$\bar{X}^r(t) = \bar{X}(t) + \frac{1}{\sqrt{r}} \hat{V}^r(t)$$

↑ random fluctuations



$$\hat{V}^r(t) = \hat{V}^r(0) + \sum_{k=1}^K (\nu_k' - \nu_k) \left[\frac{N_k(r) \int_0^t f_k^r(\bar{X}^r(s)) ds - r \int_0^t f_k^r(\bar{X}^r(s)) ds}{\sqrt{r}} \right]$$

denote:

$$+ \left[\sqrt{r} \int_0^t (f_k^r(\bar{X}^r(s)) - f_k(\bar{X}^r(s))) ds \right] \text{ I, } \eta_k^r(\cdot) \Rightarrow 0$$

$$+ \left[\sqrt{r} \int_0^t (f_k(\bar{X}^r(s)) - f_k(\bar{X}(s))) ds \right] \text{ II}$$

Denote $\bar{\Phi}_k^r(t) = \int_0^t f_k^r(\bar{X}^r(s)) ds$

Cont.

Also ~~we know~~: $\hat{N}_k^r(t) = \frac{N_k(r, t) - r t}{\sqrt{r}}$

$$\text{cont.} = \hat{V}^r(\omega) + \sum_{k=1}^K (\nu_k' - \nu_k) \left[\hat{N}_k^r(\bar{\Phi}_k^r(t)) + \sqrt{r} \int_0^t (f_k^r(\bar{X}^r(s)) - f_k(\bar{X}(s))) ds + \sqrt{r} \int_0^t (f_k(\bar{X}^r(s)) - f_k(\bar{X}(s))) ds \right] \quad (2)$$

Showed last time that

$$(\bar{\Phi}_1^r, \dots, \bar{\Phi}_k^r) \Rightarrow (\bar{\Phi}_1, \dots, \bar{\Phi}_k) \quad \text{where} \quad \bar{\Phi}_k(t) = \int_0^t f_k(\bar{X}(s)) ds$$

$$(\hat{N}_1^r, \dots, \hat{N}_k^r) \Rightarrow (W_1, \dots, W_k) \quad \text{as} \quad r \rightarrow \infty$$

where $W_i, i=1, \dots, k$ are indepen. standard 1-D Brownian motion

By Random time change Then

$$(\hat{N}_1^r(\bar{\Phi}_1^r(\cdot)), \dots, \hat{N}_k^r(\bar{\Phi}_k^r(\cdot))) \Rightarrow (W_1(\bar{\Phi}_1(\cdot)), \dots, W_k(\bar{\Phi}_k(\cdot)))$$

for $I = \sqrt{r} \int_0^t (f_k^r(\bar{X}^r(s)) - f_k(\bar{X}(s))) ds$

know from tightness proof of last time.

that for each $\varepsilon > 0, T > 0, \exists k_1, r_1$

$$\text{s.t. } P(\|\bar{X}^r(\cdot)\|_T > k_1) < \varepsilon \quad \forall r > r_1$$

on any compact set
↓
 $\frac{1}{r} g_r(x)$

$$f_k^r(x) = \prod_{i=1}^n x_i \left(x_i - \frac{1}{r}\right) \dots \left(x_i - \frac{V_{ik-1}}{r}\right) = f_k(x) + O\left(\frac{1}{r}\right)$$

$$\text{Hence } \exists k_2, r_2, \text{ s.t. } P\left(\|f_k^r(\bar{X}^r(\cdot)) - f_k(\bar{X}^r(\cdot))\|_T > \frac{k_2}{r}\right) < \varepsilon \quad \forall r \geq r_2, k=1, \dots, K$$

$$\left| \sqrt{r} \int_0^t (f_k^r(\bar{X}^r(s)) - f_k(\bar{X}^r(s))) ds \right| \leq \sqrt{r} t \cdot \frac{k_2}{r} \leq \frac{k_2 T}{\sqrt{r}} \quad \left[\begin{array}{l} \text{On} \\ \{ \|f_k^r(\bar{X}^r(\cdot)) - f_k(\bar{X}^r(\cdot))\|_T \leq \frac{k_2}{r} \} \\ \text{for all } t \in [0, T] \end{array} \right]$$

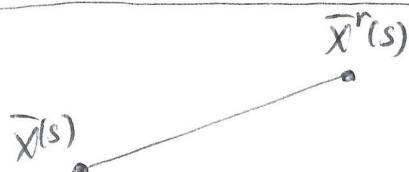
$$\Rightarrow P\left(\left\| \sqrt{r} \int_0^\cdot (f_k^r(\bar{X}^r(s)) - f_k(\bar{X}^r(s))) ds \right\|_T > \frac{k_2 T}{\sqrt{r}}\right) < \varepsilon \quad \forall r > r_2$$

$$\Rightarrow \sqrt{r} \int_0^\cdot (f_k^r(\bar{X}^r(s)) - f_k(\bar{X}^r(s))) ds \Rightarrow 0 \quad \text{as } r \rightarrow \infty$$

$$\Rightarrow I \Rightarrow 0 \quad \text{as } r \rightarrow \infty$$

for II

(3)



$$f_k(\bar{x}(s) + t(\bar{x}^r(s) - \bar{x}(s)))$$

$$t \in [0, 1]$$

$$= f_k(\bar{x}(s)) + \int_0^1 \nabla f_k(\bar{x}(s) + u(\bar{x}^r(s) - \bar{x}(s))) \cdot (\bar{x}^r(s) - \bar{x}(s)) du$$

MVT

$$\nabla f_k(\bar{x}(s) + \tilde{u}(\bar{x}^r(s) - \bar{x}(s))) \cdot (\bar{x}^r(s) - \bar{x}(s))$$

↑
some where on the line $\rho^r(s)$

$$\text{Let: } \mathcal{A}_k^r(t) = \sqrt{r} \int_0^t (f_k(\bar{x}^r(s)) - f_k(\bar{x}(s))) ds$$

$$= \sqrt{r} \int_0^t \nabla f_k(\rho_k^r(s)) \cdot (\bar{x}^r(s) - \bar{x}(s)) ds \quad \text{where}$$



$$\rho_k^r(s) \in [\bar{x}(s), \bar{x}^r(s)]$$

$$= \int_0^t \nabla f_k(\rho_k^r(s)) \cdot \hat{V}^r(s) ds$$

$$\therefore \hat{V}^r(t) = \hat{V}^r(0) + \sum_{k=1}^K (\nu_k' - \nu_k) [\hat{N}_k^r(\bar{\Phi}_k^r(t)) + \eta_k^r(t) + \int_0^t \nabla f_k(\rho_k^r(s)) \cdot \hat{V}^r(s) ds]$$

Since have compact containment for $\{\bar{x}^r\}$,

given $\Sigma > 0 \quad \exists K^+, r^+, s.t.$

$$P(\|\rho_k^r(\cdot)\|_T \geq K^+) < \Sigma \quad \forall r \geq r^+$$

Since ∇f_k is cts for each k it follows that

$$\exists K^+ s.t. \quad P(\|\nabla f_k(\rho_k^r(\cdot))\|_T \geq K^+) < \Sigma \quad \forall r \geq r^+$$

To prove $\{\hat{V}^r\}$ is C-tight (Assume $\hat{V}^r(0) \Rightarrow \hat{V}(0)$)

For each $T > 0 \quad \Sigma > 0 \quad \exists K^+, r^+ s.t. \quad \forall r \geq r^+$

$$P(\|\hat{V}^r(0) + \sum_{k=1}^K (\nu_k' - \nu_k) [\hat{N}_k^r(\bar{\Phi}_k^r(\cdot)) + \eta_k^r(t)]\| \geq K^+, \left. \begin{array}{l} \|\nabla f_k(\rho_k^r(\cdot))\|_T \leq \frac{K^+}{(\nu_k' - \nu_k + 1)K}, k=1, \dots, K \end{array} \right) \geq 1 - \Sigma \quad \leftarrow G_r$$

① opt
containment
for $\{\hat{V}^r\}$

(4)

on Good set G^n

$$|\hat{V}^n(t)| \leq K^* + \sum_{k=1}^K (V_k' - V_k) \int_0^t \frac{K^*}{(V_k' - V_k + 1)k} |\hat{V}^n(s)| ds$$

$$\leq K^* + K^* \int_0^t |\hat{V}^n(s)| ds$$

Gronwall
ineq.

$$\leq K^* + K^* (e^{K^* t} - 1) \quad t \in [0, T]$$

$$\Rightarrow \exists K^{**} \text{ st. } P(\|\hat{V}^n(\cdot)\|_T \leq K^{**}) \geq 1 - \varepsilon \quad \forall n \geq n^*$$

(2) Control the oscillations

On G^n $\left\{ \int_0^t \nabla f_k(p_k^n(s)) \cdot \hat{V}^n(s) ds, t \in [0, T] \right\}$ is uniformly Lipschitz continuousGives sufficient control on oscillations of $X^n(\cdot)$ $\Rightarrow \{\bar{X}^n(\cdot)\}_{n \geq 0}$ are C -tight $\Rightarrow \{\hat{V}^n(\cdot)\}$ is C -tightSuppose $\hat{V}$ is a weak limit point. $[p_k^n(s) \in [\bar{X}(s), \bar{X}^n(s)]]$
 $n \rightarrow \infty$

$$\text{Then } \hat{V}(t) = \hat{V}(0) + \sum_{k=1}^K (V_k' - V_k) \left[W_k \left(\int_0^t f_k(\bar{X}(s)) ds \right) + \int_0^t \nabla f_k(\bar{X}(s)) \cdot \hat{V}(s) ds \right]$$

$$= \hat{V}_0(0) + \sum_{k=1}^K (V_k' - V_k) \left[G_k(t, \omega) + \int_0^t H_k(s, \omega) \cdot \hat{V}(s) ds \right]$$

 $\uparrow$ (linear in $\hat{V}$)Given $\hat{V}(0) \exists!$ soln $\hat{V}$ of this equationFollows that $\hat{V}^n \Rightarrow \hat{V}$

$$\bar{X}^n(t) \leq \bar{X}(t) + \frac{1}{\sqrt{n}} \hat{V}(t)$$

MUST KNOW

(5)

Gronwall's inequality

Suppose f, g are non-negative Lebesgue integrable functions on $[0, T]$ and:

$$f(t) \leq g(t) + c \int_0^t f(s) ds \quad \forall t \in [0, T], \quad c > 0$$

$$\text{Then } f(t) \leq g(t) + c \int_0^t e^{c(t-s)} g(s) ds$$

pf Let $V(t) = \int_0^t f(s) ds$ a.e.-t. $\frac{dV}{dt} = f(t) \leq g(t) + cV(t)$

$$\Rightarrow e^{-ct} \frac{dV}{dt} \leq e^{-ct} g(t) + (e^{-ct} V(t))$$

$$e^{-ct} \frac{dV}{dt} - (e^{-ct} V(t)) \leq e^{-ct} g(t)$$

$$\frac{d(e^{-ct} V)}{dt} \leq e^{-ct} g(t) \quad \text{Integrate}$$

$$e^{-ct} V(t) - V(0) \leq \int_0^t e^{-cs} g(s) ds$$

$$V(0) = 0$$

$$V(t) \leq \int_0^t e^{-c(t-s)} g(s) ds$$

$$f(t) \leq g(t) + cV(t)$$

$$\leq g(t) + c \int_0^t e^{-c(t-s)} g(s) ds$$