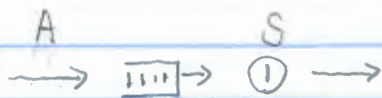


$G|G|1$ FIFO (First In, First Out)



① Let arrival A be a renewal process

associated with i.i.d sequence $\{u_i\}_{i=1}^{\infty}$

where u_i is interarrival time

with mean $1/\lambda$, variance σ_a^2

② Let S be a renewal process with i.i.d sequence $\{v_i\}_{i=1}^{\infty}$

v_i is the ^{ith} service time

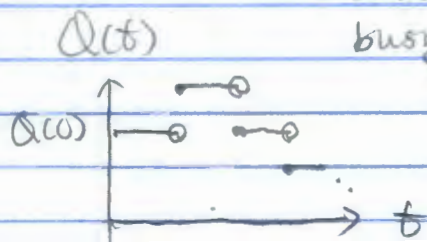
with mean $1/\mu$, variance σ_s^2

③ $\{u_i\}_{i=1}^{\infty}$ and $\{v_i\}_{i=1}^{\infty}$ are independent.

④ $Q(t) = \#$ customers in system at time t .

$= Q(0) + A(t) - D(t)$ ← cumulative departure up to time t

$= Q(0) + A(t) - S(T(t))$ where $T(t) = \int_0^t \mathbf{1}_{\{Q(s) > 0\}} ds$
 $=$ cumulative busy time.



page 2

⑤ $W(t)$ = Workload process

$$= W(0) + \sum_{i=1}^{A(t)} V_i - T(t)$$

$$= W(0) + \sum_{i=1}^{A(t)} V_i - t + I(t), \text{ where}$$

$I(t)$ = cumulative idle time

$$= t - T(t)$$

Note $I(t)$ increases only when $W=0$.

By Functional Central Limit Theorem (FCLT)

$$\left\{ \frac{A(nt) - \lambda nt}{\lambda^{3/2} \sigma_a \sqrt{n}}, t \geq 0 \right\} \Rightarrow B_a$$

Similarly,

$$\left\{ \frac{S(nt) - \mu nt}{\mu^{3/2} \sigma_s \sqrt{n}}, t \geq 0 \right\} \Rightarrow B_s$$

Note B_a, B_s are independent Brownian Motions.

By Functional Law of Large Numbers (follows from FCLT)

$$\left\{ \frac{A(nt) - \lambda nt}{n}, t \geq 0 \right\} \Rightarrow 0, \text{ so } \left\{ \frac{A(nt)}{n}, t \geq 0 \right\} \Rightarrow \lambda(\cdot)$$

Similarly,

$$\lambda(t) = \lambda t, t \geq 0$$

$$\left\{ \frac{S(nt)}{n}, t \geq 0 \right\} \Rightarrow \mu(\cdot) \quad \mu(t) = \mu t, t \geq 0.$$

Page 3

$$Q(t) = Q(0) + A(t) - \int_0^t T(s) ds. \quad \text{Assume } Q$$

Consider a sequence of systems indexed by n , where $Q(0)$ is allowed to change with n .
Fluid Limit.

$$\bar{Q}^n(t) \stackrel{\text{def}}{=} Q^n(nt)/n. \quad \text{Assumes } \bar{Q}^n(0) \text{ is indep of } A, S.$$

$$\bar{A}^n(t) \stackrel{\text{def}}{=} A^n(nt)/n.$$

$$\bar{Q}^n(0) \Rightarrow \bar{q}(0)$$

Here A^n, S^n do not depend on n

$$\bar{S}^n(t) \stackrel{\text{def}}{=} S^n(nt)/n.$$

$$\bar{T}^n(t) \stackrel{\text{def}}{=} T^n(nt)/n$$

then

$$\bar{Q}^n(t) = \bar{Q}^n(0) + \bar{A}^n(t) - \int_0^t \bar{T}^n(s) ds.$$

and

$$\bar{T}^n(t) = \int_0^{nt} \mathbb{1}_{\{Q^n(s) > 0\}} ds / n$$

$$= \int_0^t \mathbb{1}_{\{Q^n(n\tilde{s}) > 0\}} d\tilde{s}, \quad \tilde{s} = s/n$$

$$= \int_0^t \mathbb{1}_{\{\bar{Q}^n(\tilde{s}) > 0\}} d\tilde{s}$$

so

$$\|\bar{T}^n\|_{[0,K]} \leq K.$$

$$w(\bar{T}^n, \delta, K) = \sup_{\substack{s, t \in [0, K] \\ |s-t| \leq \delta}} |\bar{T}^n(s) - \bar{T}^n(t)| \leq \delta.$$

Then, $\{\bar{T}^n\}$ ~~are~~ ^{is} C -tight.

page 4

Therefore,

$(\bar{A}^n, \bar{S}^n, \bar{T}^n) \Rightarrow C\text{-tight}$ because
each component is $C\text{-tight}$. + can include $\bar{Q}^n$ because $\bar{Q}^n(0)$
converges in distr.

Then Consider any "weak" point

$$(\bar{Q}, (\bar{A}, \bar{S}, \bar{T}), \text{ or } (\bar{Q}^n, \bar{A}^n, \bar{S}^n, \bar{T}^n), \text{ i.e. } \exists \text{ subsequence } \{n_k\} \text{ of } \{n\} \text{ st.}$$
$$(\bar{Q}^n, (\bar{A}^{n_k}, \bar{S}^{n_k}, \bar{T}^{n_k})) \Rightarrow (\bar{Q}, (\bar{A}, \bar{S}, \bar{T})) \text{ as } k \rightarrow \infty.$$
$$\quad \quad \quad \parallel \quad \parallel$$
$$\quad \quad \quad \lambda(\cdot) \quad \mu(\cdot).$$

Hence.

$$\bar{Q}^n(t) = \bar{Q}^n(0) + \bar{A}^n(t) - \bar{S}^n(\bar{T}^n(t))$$

$$\Downarrow \quad \quad \Downarrow \quad \quad \Downarrow$$
$$\bar{Q}(t) = \bar{Q}(0) + \lambda(t) - \mu \bar{T}(t).$$

Note,

$$\text{because } (\bar{S}^{n_k}, \bar{T}^{n_k}) \Rightarrow (\mu(\cdot), \bar{T}(\cdot))$$

(by Random Time Change Theorem)

$$\Rightarrow \bar{S}^n(\bar{T}^{n_k}(\cdot)) \Rightarrow \mu(\bar{T}(\cdot)) = \mu \bar{T}(\cdot).$$