

Chapter 3

Skorokhod Problems

When a service station in a queueing network is empty, the jobcount and workload processes enter the boundaries of their state spaces. Such boundary effects are typical in queueing networks and are one of the main sources of difficulty in their analysis. In this chapter we introduce the Skorokhod problem (SP) which is a key tool for addressing such boundary problems.

For a positive integer d , recall that $\mathbb{D}^d$ is the set of functions from $[0, \infty)$ into $\mathbb{R}^d$ that are right continuous on $[0, \infty)$ and have finite left limits in $(0, \infty)$. Such a function is also called a *path*. We use $\mathbb{C}^d$ to denote the subset of continuous paths. For $x \in \mathbb{D}^d$, we let $x(t-)$ denote the left limit for x at $t > 0$ and we define $x(0-)$ to equal $x(0)$. For each $t \geq 0$, we let $\Delta x(t) = x(t) - x(t-)$, the jump of x at t .

We start the chapter with one-dimensional SPs, for which the existence and uniqueness of solutions are proved. The major part of this chapter focuses on multidimensional SPs in an orthant. Existence and uniqueness in this case are not guaranteed, but depend on the properties of a so-called reflection matrix. Care has been taken to make the SPs cover paths with jumps, not just continuous paths. The inclusion of paths with jumps makes our presentation more complicated than it would be for continuous paths alone, however, it is needed for our applications to queueing networks. Finally, we extend the SPs to general polyhedrons. Such SPs arise in networks with finite buffers and in closed queueing networks.

3.1 One-Dimensional Skorokhod Problem

In this section, we define the notion of a solution of the one-dimensional Skorokhod problem. We show that the solution exists and is unique. This is used to define a path-to-path mapping called the reflection map, or Skorokhod map. We show that the reflection map has a Lipschitz continuity property with respect to the uniform norm over finite time intervals. We present a single server system in which the Skorokhod problem arises naturally.

When $d = 1$, the path spaces $\mathbb{D}^d$ and $\mathbb{C}^d$ are simply denoted by $\mathbb{D}$ and $\mathbb{C}$. Let $\mathbb{D}_+ = \{x \in \mathbb{D} : x(0) \geq 0\}$. For a path $x \in \mathbb{D}_+$, we now define a solution $(z, y) \in \mathbb{D}_+ \times \mathbb{D}_+$ of the Skorokhod problem (SP) for x .

Definition 3.1.1 (One-Dimensional Skorokhod Problem). Let $x \in \mathbb{D}_+$. A pair $(z, y) \in \mathbb{D}_+ \times \mathbb{D}_+$ is a solution of the (one-dimensional) Skorokhod problem for x if the following hold:

- (i) $z(t) = x(t) + y(t), t \geq 0$,
- (ii) $z(t) \geq 0, t \geq 0$,
- (iii) y satisfies the following:
 - (a) $y(0) = 0$,
 - (b) y is nondecreasing,
 - (c) $\int_{(0,\infty)} z(t) dy(t) = 0$.

The solution (z, y) is also called a *regulation* of x , where y is a *regulator* and z is a *regulated path*. In general, the path x can be negative at some time points. Condition (ii) requires that the regulated path, z , be everywhere nonnegative. Equation (i) states that the regulation is obtained by adding a regulator, y , to the path x . The regulator is required to start from 0 and be nondecreasing. The integral in (iii c) is to be interpreted in the Lebesgue-Stieltjes sense. The following lemma provides an equivalent characterization of condition (iii c).

Lemma 3.1.2. Suppose that a pair $(z, y) \in \mathbb{D}_+ \times \mathbb{D}_+$ is such that (ii), (iii a) and (iii b) hold. Then, condition (iii c) is equivalent to the following: for each $0 < s \leq t$,

$$y(t) = y(s-) \quad \text{whenever } z(u) > 0 \text{ for each } u \in [s, t]. \quad (3.1)$$

Condition (3.1) yields the following interpretation for (iiic): $y(\cdot)$ does not increase in an interval $[s, t]$ whenever $z(\cdot)$ is positive in the interval.

There is another equivalent condition for (iiic) that we now state. For this, we decompose $y \in \mathbb{D}_+$ satisfying (iiia) and (iiib) into its continuous part y^c and its pure jump part y^d . More precisely, for each $t \geq 0$, let $y^d(t) = \sum_{0 < s \leq t} \Delta y(s)$ and $y^c(t) = y(t) - y^d(t)$. The sum in the definition of y^d has only countably many non-zero terms and each of these is strictly positive since y has only countable many jumps and it is nondecreasing. Both y^c and y^d are nondecreasing and start from 0.

Lemma 3.1.3. *Suppose that a pair $(z, y) \in \mathbb{D}_+ \times \mathbb{D}_+$ is such that (ii), (iiia) and (iiib) hold. Then, condition (iiic) is equivalent to the following:*

$$z(t) = 0 \text{ for each } t > 0 \text{ satisfying } \Delta y(t) > 0, \text{ and for each } 0 < s \leq t, \quad (3.2)$$

$$y^c(t) = y^c(s) \text{ whenever } z(u) > 0 \text{ for each } u \in [s, t]. \quad (3.3)$$

Both Lemma 3.1.2 and Lemma 3.1.3 will be proved at the end of this section.

We will prove shortly that for each $x \in \mathbb{D}_+$, there exists a solution (z, y) of the Skorokhod problem for x and that the solution is unique. This allows us to define the (one-dimensional) *reflection map* (or Skorokhod map), $(\phi, \psi) : \mathbb{D}_+ \rightarrow \mathbb{D}_+ \times \mathbb{D}_+$ such that

$$(\phi(x), \psi(x)) = (z, y), \quad (3.4)$$

where (z, y) is the unique solution of the Skorokhod problem for $x \in \mathbb{D}_+$. Before we prove the existence and uniqueness, we first provide an example in which the Skorokhod problem arises naturally.

Example 3.1.4 (Workload in a Single Server System). Consider a single server queueing system that starts empty and in which the server follows a non-idling policy. Suppose that we are interested in the workload process $W = \{W(t), t \geq 0\}$, where $W(t)$ is the workload for the server at time t . Recall that the workload at time t is defined to be the amount of time that the server has to work in order to empty out the system when the arrivals are turned off at time t . It is also called the virtual waiting time at time t . (The reason for this terminology is that if an arrival were to occur at time t , this arrival would have to wait $W(t)$ units of time before it entered service under the FIFO service policy.) As in the previous chapter, for each time $t \geq 0$, $E(t)$ denotes the number of arrivals to the system by time t and $v(i)$ denotes the service time required by the i th arrival. The workload process has the following representation which is convenient for obtaining fluid and diffusion approximations

of the process; these approximations are discussed in later chapters. For each $t \geq 0$,

$$W(t) = \sum_{i=1}^{E(t)} v(i) - t + Y(t), \quad (3.5)$$

where $Y(t)$ is the cumulative amount of time that the server has been idle in $[0, t]$. To understand (3.5), note that $\sum_{i=1}^{E(t)} v(i)$ is the total workload input by time t . The server could have deleted t units of work if the server had never been idle in $[0, t]$. Thus,

$$X(t) = \sum_{i=1}^{E(t)} v(i) - t \quad (3.6)$$

would be the workload at time t if the server were never idle in $[0, t]$. For this reason, $X = \{X(t), t \geq 0\}$ is sometimes called the *netput* process. Since, by time t , $Y(t)$ units of server time is lost, the workload $W(t)$ is equal to $X(t) + Y(t)$. It is clear that the workload is nonnegative, and the idletime process Y is nondecreasing and starts from 0. The non-idling service policy requires that the server should never idle when there is work to be done. This, together with the fact that the sample paths of Y are continuous (being measurements of time), implies via Lemma 3.1.2 that condition (iiic) holds with a sample path of (X, W, Y) in place of (x, z, y) . It follows that sample path by sample path, (W, Y) is a regulation of the netput process X . Thus, using the reflection map defined in (3.4), we have

$$W = \phi(X) \quad \text{and} \quad Y = \psi(X).$$

Now we present the existence and uniqueness statement for the Skorokhod problem.

Theorem 3.1.5. *Let $x \in \mathbb{D}_+$. Then there exists a unique solution $(z, y) \in \mathbb{D}_+ \times \mathbb{D}_+$ of the SP for x , given by*

$$y(t) = \sup_{0 \leq s \leq t} x^-(s), \quad t \geq 0, \quad (3.7)$$

$$z(t) = x(t) + y(t), \quad t \geq 0, \quad (3.8)$$

where $x^-(t) = \max\{-x(t), 0\}$. Furthermore, if x is continuous, then both y and z are continuous.

Proof. When x is continuous, the proof is simpler. On a first pass, the reader may wish to restrict to this case.

We first prove the uniqueness. Suppose that there are two solutions (z, y) and $(\widehat{z}, \widehat{y})$ for x . Then $z - \widehat{z} = y - \widehat{y}$. Now let $f = y - \widehat{y}$. Note that $f(0) = 0$. By Lemma A.7.1, we have for each $t \geq 0$,

$$\begin{aligned}
0 &\leq f^2(t) + \sum_{0 \leq s \leq t} (\Delta f(s))^2 \\
&= 2 \int_{(0,t]} f(s) df(s) \\
&= 2 \int_{(0,t]} (z(s) - \widehat{z}(s)) d(y(s) - \widehat{y}(s)) \\
&= -2 \int_{(0,t]} \widehat{z}(s) dy(s) - 2 \int_{(0,t]} z(s) d\widehat{y}(s) \\
&\leq 0,
\end{aligned}$$

where we have used condition (iiic) for (z, y) and $(\widehat{z}, \widehat{y})$ to obtain the third equality. Hence $f(t) = 0$, for each $t \geq 0$, which proves the desired uniqueness.

For the proof of existence, let $y(t) = \sup_{0 \leq s \leq t} x^-(s)$ for each $t \geq 0$. Since $x(0) \geq 0$, $x^-(0) = 0$ and so $y(0) = 0$. Clearly,

$$z(t) \equiv x(t) + y(t) \geq x(t) + x^-(t) \geq 0 \quad \text{for all } t \geq 0,$$

and y is nondecreasing, so it has left limits on $(0, \infty)$. Since x is right continuous, y is right continuous. To complete the proof of existence, it remains to verify that y satisfies property (iiic) in the definition of the Skorokhod problem.

For this, first consider a $t > 0$ such that $\Delta y(t) > 0$. Then, since $\Delta y(t) = y(t) - y(t-)$, $y(t) > y(t-)$ and so $x^-(t) > \sup_{s \in [0,t)} x^-(s) \geq 0$. It follows that $y(t)$ must equal $x^-(t) = -x(t)$ and so $z(t) = x(t) + y(t) = x(t) - x(t) = 0$. Thus, we have proved

$$\Delta y(t) > 0 \text{ for some } t > 0 \text{ implies } z(t) = 0. \quad (3.9)$$

Now, suppose that $0 < s \leq t$ and $y(t) > y(s-)$. We aim to show that $z(u) = 0$ for some $u \in [s, t]$. If $\Delta y(u) > 0$ for some $u \in [s, t]$, then by (3.9), $z(u) = 0$. Consequently, it suffices to treat the case where y is continuous on $[s, t]$ with $y(s) = y(s-)$. Assuming this, we will show that there is $u \in [s, t]$ such that $y(u) = -x(u)$, from which it follows that $z(u) = x(u) + y(u) = 0$. For this, let $u^* = \inf\{u > s : y(u) > y(s)\}$. Then $u^* \in [s, t]$, since $y(t) - y(s) > 0$, and we have $y(u) = y(s)$ for all $u \in [s, u^*]$. (Here we have used the assumption that y is continuous on $[s, t]$.) From the cumulative nature of y , for each $u \geq u^*$, $y(u) = y(u^*) \vee \sup\{-x(v) : v \in [u^*, u]\}$. Combining this with the definition of u^* , we see that for each $\epsilon > 0$ there is $v_\epsilon \in [u^*, u^* + \epsilon]$ such

that $y(s) = y(u^*) \leq -x(v_\epsilon)$. On letting $\epsilon \rightarrow 0$, we obtain by the right continuity of x that $y(u^*) \leq -x(u^*)$. But since $y(u^*) \geq -x(u^*)$, by definition, it follows that $y(u^*) = -x(u^*)$.

Finally, when x is continuous, it can be checked that $y(t) = \sup_{0 \leq s \leq t} x^-(s)$ is a continuous function of t , and hence so is $z(t) = x(t) + y(t)$. $\square$

We next prove that the reflection map has a Lipschitz continuity property with respect to the uniform norm over finite time intervals. Recall that for $x \in \mathbb{D}$ and $t \geq 0$, $\|x\|_t = \sup_{0 \leq s \leq t} |x(s)|$. Usually the space $\mathbb{D}$ is endowed with the Skorokhod J_1 -topology, instead of the topology induced by the uniform norm. However, the following result plays a role in investigating continuity of the reflection map in the J_1 -topology. We defer discussion of the latter to the next section, where the multidimensional Skorokhod problem is discussed.

Theorem 3.1.6. *For $x, \tilde{x} \in \mathbb{D}_+$ and $t \geq 0$,*

$$\|\psi(x) - \psi(\tilde{x})\|_t \leq \|x - \tilde{x}\|_t, \quad (3.10)$$

$$\|\phi(x) - \phi(\tilde{x})\|_t \leq 2 \|x - \tilde{x}\|_t. \quad (3.11)$$

Proof. The second inequality follows from the first inequality and (3.8). The first inequality follows from the explicit expression for $\psi(x)$ in terms of x (cf. (3.7)) and the fact that for any $w, \tilde{w} \in \mathbb{D}$ and $t \geq 0$,

$$\left| \sup_{0 \leq s \leq t} w(s) - \sup_{0 \leq s \leq t} \tilde{w}(s) \right| \leq \sup_{0 \leq s \leq t} |w(s) - \tilde{w}(s)|,$$

and the fact that for any $a, b \in \mathbb{R}$,

$$|a^- - b^-| \leq |a - b|.$$

$\square$

We end this section with proofs of Lemmas 3.1.2 and 3.1.3.

Proof of Lemma 3.1.2. We first prove that equation (iiic) implies condition (3.1). Let $0 < s \leq t$ be fixed. Assume that $z(u) > 0$ for $u \in [s, t]$. We are to prove $y(t) = y(s-)$. For each integer $n \geq 1$,

$$\int_{[s,t]} z(u) dy(u) \geq \int_{[s,t]} z(u) 1_{\{u: z(u) > 1/n\}} dy(u) \geq n^{-1} \int_{[s,t]} 1_{\{u: z(u) > 1/n\}} dy(u).$$

Since (iii)(c) implies $\int_{[s,t]} z(u) dy(u) = 0$,

$$\int_{[s,t]} 1_{\{u: z(u) > 1/n\}} dy(u) = 0 \text{ for each } n.$$

Letting $n \rightarrow \infty$, by the monotone convergence theorem,

$$\int_{[s,t]} 1_{\{u: z(u) > 0\}} dy(u) = 0.$$

Since $z(u) > 0$ for $u \in [s, t]$,

$$y(t) - y(s-) = \int_{[s,t]} 1_{\{u: z(u) > 0\}} dy(u) = 0.$$

Now, we prove that condition (3.1) implies (iiic). Since $z(t) \geq 0$ for all $t \geq 0$ and y is nondecreasing, condition (iiic) holds if and only if the following two equalities hold:

$$\int_{(0,\infty)} z(t) dy^c(t) = 0, \quad (3.12)$$

$$\int_{(0,\infty)} z(t) dy^d(t) = 0. \quad (3.13)$$

We prove (3.13) first. Suppose that (3.13) does not hold. By definition,

$$\int_{(0,\infty)} z(t) dy^d(t) = \sum_{0 < t < \infty} z(t) \Delta y(t). \quad (3.14)$$

Thus, there exists $t > 0$ such that $z(t) \Delta y(t) > 0$, which contradicts condition (3.1) with $s = t$. Thus, (3.13) holds.

Now, turning to (3.12), since y^c is continuous and non-decreasing, and z is bounded on each compact time interval, for each $t \geq 0$, $\int_{(0,t]} z(s) dy^c(s)$ is a Riemann-Stieltjes integral (cf. Graves (1956, p. 278)). Hence, we can use lower Darboux sums to approximate it (cf. Graves (1956, p. 269–270)). In particular, for each $t \geq 0$,

$$\int_{(0,t]} z(s) dy^c(s) = \lim_{n \rightarrow \infty} \sum_{k=1}^{2^n} \left(\inf_{s \in [t_{k-1}^n, t_k^n]} z(s) \right) (y^c(t_k^n) - y^c(t_{k-1}^n)), \quad (3.15)$$

where $t_k^n = kt/2^n$ for $k = 0, 1, \dots, 2^n$ and $n = 1, 2, \dots$. If (3.1) holds, then for each $t \geq 0$, the right member of (3.15) is zero and hence so is the left member. It then follows that (3.12) holds, since t was arbitrary. $\square$

Proof of Lemma 3.1.3. The proof is similar to the proof of Lemma 3.1.2. We give an outline here. Assume that condition (iiic) holds. Condition (3.2) follows from (3.13) and (3.14) in the proof of Lemma 3.1.2. To prove (3.3), note that condition (iiic) implies (3.12). Thus, condition (3.3) follows from the first part of the proof of Lemma 3.1.2.

Now suppose that conditions (3.2) and (3.3) hold. It follows from the second part of the proof of Lemma 3.1.2 that condition (iiic) holds. $\square$

3.2 Skorokhod Problem in an Orthant

The multidimensional Skorokhod problem in an orthant $\mathbb{R}_+^d$, for some positive integer d , can be defined analogously to the one-dimensional case. There is an important difference, however. A $d \times d$ matrix R , later called the reflection matrix, needs to be given as part of the problem specification. The existence and uniqueness of solutions of the multidimensional SPs depend on the properties of the matrix R . Sufficient conditions on R for the existence and uniqueness will be given in Section 3.3.

In this section, we first present the definition of the Skorokhod Problem associated with a reflection matrix R . Throughout this section, vector inequalities are interpreted componentwise. Also, d is a fixed positive integer, R is a fixed $d \times d$ matrix, and $\mathbb{D}_+^d$ denotes the set of paths $x \in \mathbb{D}^d$ satisfying $x(0) \geq 0$.

Definition 3.2.1 (Multi-dimensional Skorokhod Problem). Let $x \in \mathbb{D}_+^d$. A pair $(z, y) \in \mathbb{D}_+^d \times \mathbb{D}_+^d$ is a solution of the Skorokhod problem for x (with respect to R) if the following conditions hold:

- (i) $z(t) = x(t) + Ry(t)$, $t \geq 0$,
- (ii) $z(t) \geq 0$, $t \geq 0$,
- (iii) y is such that for $i = 1, \dots, d$,
 - (a) $y_i(0) = 0$,
 - (b) y_i is nondecreasing,
 - (c) $\int_{(0, \infty)} z_i(t) dy_i(t) = 0$.

A solution (z, y) is called an R -regulation for x . The path y is a *regulator* for x and z a *regulated* path for x . In the definition, the path x may go outside of the orthant

at some time points. Condition (ii) requires that the regulated path is always inside the orthant. Let v^i denote the i th column of R . Condition (i) is equivalent to

$$z(t) = x(t) + \sum_{i=1}^d v^i y_i(t), \quad t \geq 0. \quad (3.16)$$

Thus, z is constrained in the orthant by employing the nondecreasing regulator $y = (y_1, \dots, y_d)$. For each i , one unit of increase of y_i contributes a displacement v_i to the path z . Condition (iiic) states that y_i increases in an interval only when z reaches the boundary $F_i = \{a \in \mathbb{R}_+^d : a_i = 0\}$ in the interval. The increase of y_i “pushes” z into the orthant in the direction of v^i . The vector v^i is called the *direction of reflection* on the face F_i , and the matrix R is called the *reflection matrix*. We use the term “reflection” for historical reasons. Its usage is misleading because there is no physical reflection in the description of the Skorokhod problem.

We hope that the R -regulation for x exists and is unique for each $x \in \mathbb{D}_+^d$. In such a case, the map $x \in \mathbb{D}_+^d \rightarrow (z, y) \in \mathbb{D}_+^d \times \mathbb{D}_+^d$ is well defined, and we let $z = \phi(x)$ and $y = \psi(x)$. The map (ϕ, ψ) is called the (oblique) *reflection map* or Skorokhod map. The existence and uniqueness depend on the reflection matrix R . When R is a generalized $\mathcal{M}$ -matrix (to be defined in Section 3.3), we will show that the reflection map is well defined. Furthermore, it is Lipschitz continuous in the uniform norm over finite time intervals. We will also show that the reflection map is continuous under the Skorokhod J_1 -topology. We end this section by providing an example in which the multidimensional Skorokhod problem arises.

Example 3.2.2 (Tandem Fluid Model). Consider two fluid queues in tandem. Let $Z_i(t)$ be the fluid level at time t in queue i , $i = 1, 2$. When the queue at station i has a positive fluid level, the server pumps fluid at rate μ_i . Fluid arrives at station 1 from outside at rate α_1 . Fluid leaving station 1 goes to station 2, and, after service there, leaves the system.

Note that $\mu_i t$ is the potential output of fluid from station i in $[0, t]$. We use $Y_i(t)$ to denote the cumulative lost output from station i in $[0, t]$. Thus, $\mu_i t - Y_i(t)$ is the actual amount of fluid that has departed station i by time t . Therefore,

$$\begin{aligned} Z_1(t) &= Z_1(0) + \alpha_1 t + (\mu_1 t - Y_1(t)), \\ Z_2(t) &= Z_2(0) + (\mu_1 t - Y_1(t)) - (\mu_2 t - Y_2(t)). \end{aligned}$$

In vector form $Z(t) = X(t) + RY(t)$, where $R = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$ and

$$X_1(t) = Z_1(0) + (\alpha_1 - \mu_1)t, \quad X_2(t) = Z_2(0) + (\mu_1 - \mu_2)t. \quad (3.17)$$

Clearly, we require $Z(t) \geq 0$ for all $t \geq 0$. It is also clear that $Y(0) = 0$ and each component of Y is nondecreasing. We require that both servers be non-idling. If the fluid level at a station is positive in the time interval $[s, t]$, there will be no output loss during the interval at the station. Thus, $\int_0^\infty Z_i(t) dY_i(t) = 0$, $i = 1, 2$. Therefore, (Z, Y) is an R -regulation for X . It turns out that R is an example of a generalized $\mathcal{M}$ -matrix. Thus, the reflection map (ϕ, ψ) is well defined, and therefore,

$$Z = \phi(X) \quad \text{and} \quad Y = \psi(X). \quad (3.18)$$

Note that our description of the tandem fluid model at the beginning of this example is an intuitive one. One can take (3.17) and (3.18) as a mathematical definition of the fluid model.

3.3 Sufficient Conditions for Existence and Uniqueness

We will first define a generalized $\mathcal{M}$ -matrix. We will show that if the reflection matrix is a generalized $\mathcal{M}$ -matrix, the reflection map is well defined. We then further show that the reflection map is Lipschitz continuous under the uniform norm over finite intervals and is continuous in the Skorokhod J_1 -topology. As before d is a positive integer. For a $d \times d$ matrix A , $|A|$ denotes the matrix whose entries are the absolute values of the entries of A , $\text{diag}(A)$ denotes the diagonal matrix whose diagonal entries are taken from those of A , and $\|A\| = \max_i \sum_j |A_{ij}|$ denotes the norm of A . For $b \in \mathbb{R}^d$, let $|b| = \max_i |b_i|$ denote the norm of b . Then, $|Ab| \leq \|A\| |b|$.

Definition 3.3.1. A $d \times d$ matrix R is said to be a generalized $\mathcal{M}$ -matrix if

- (i) each diagonal entry of R is positive, and
- (ii) the spectral radius of $|Q|$ is less than one,

where Q is the $d \times d$ matrix that satisfies

$$R = (I - Q) \text{diag}(R). \quad (3.19)$$

Remark. A generalized $\mathcal{M}$ -matrix is sometimes called a generalized Harrison-Reiman type matrix. Harrison and Reiman first studied the multidimensional Skorokhod problems when R satisfying (i) and (ii) in Definition 3.3.1 and each entry of Q in (3.19) is nonnegative and R . Such a matrix R is called a *non-singular* $\mathcal{M}$ -matrix in Berman and Plemmons [Berman and Plemmons \(1979\)](#).

The following is the main theorem of this section.

Theorem 3.3.2. *Assume that the $d \times d$ matrix R is a generalized $\mathcal{M}$ -matrix. Then for each $x \in \mathbb{D}_+^d$, an R -regulation for x exists and is unique. Thus, the reflection map (ϕ, ψ) associated with R is well defined.*

Proof. Let $x \in \mathbb{D}_+^d$ be fixed. Since each diagonal entry of R is positive, one can easily check that (z, y) is an R -regulation for x if and only if $(z, \text{diag}(R)y)$ is an $R(\text{diag}(R))^{-1}$ -regulation for x . For notational simplicity, we assume that $\text{diag}(R) = I$. Let $Q = I - R$. Since $|Q|$ has spectral radius less than one, by the result of Exercise 3.3.3, there exists a diagonal matrix B with positive diagonal entries, such that $\tilde{Q} = BQB^{-1}$ satisfies $\|\tilde{Q}\| < 1$. Also, (z, y) is an R -regulation for x if and only if (Bz, By) is an $(I - \tilde{Q})$ -regulation for Bx . Thus, without loss of generality, we assume that $\|Q\| < 1$.

Let

$$\mathbb{D}_0^d = \{y \in \mathbb{D}^d : y \text{ is nondecreasing and } y(0) = 0\}.$$

Define $\pi : \mathbb{D}_0^d \rightarrow \mathbb{D}_0^d$ by

$$\pi(y)(t) = \sup_{0 \leq s \leq t} (x(s) - Qy(s))^- , \quad t \geq 0, \quad (3.20)$$

where the negative part and the supremum are taken componentwise.

We now claim that (z, y) is an $(I - Q)$ -regulation for x if and only if (z, y) satisfies the following three properties:

$$y \in \mathbb{D}_0^d, \quad (3.21)$$

$$y = \pi(y), \quad (3.22)$$

$$z = x + (I - Q)y. \quad (3.23)$$

Note that (3.23) is just a restatement of property (i) of Definition 3.2.1 and (3.21) corresponds to (iiia) and (iiib) of Definition 3.2.1. If (z, y) is an $(I - Q)$ -regulation for x , then by the uniqueness of the solution of the one-dimensional Skorokhod problem, applied component by component to $z = (x - Qy) + y$, we must have $y = \pi(y)$. Conversely, if $y = \pi(y)$ and properties (3.21), (3.23) hold, then it follows from Theorem 3.1.5 for the one-dimensional Skorokhod problem (applied componentwise) that (ii) and (iiic) of Definition 3.2.1 hold.

Now we use a contraction argument to show that there is a unique y satisfying (3.21)-(3.23). Let $T > 0$ be fixed for the moment. By restricting functions to the time interval $[0, T]$, we can view π as a map from $\mathbb{D}_0^d[0, T]$ into $\mathbb{D}_0^d[0, T]$, where $\mathbb{D}_0^d[0, T]$ is

the set of nondecreasing functions from $[0, T]$ to $\mathbb{R}^d$ that are right continuous in $[0, T)$ and have left limits in $(0, T]$. For $y, y' \in \mathbb{D}_0^d[0, T]$ and $i = 1, \dots, d$, let $\pi_i(y)$ and $\pi_i(y')$ denote the i th component of $\pi(y)$ and $\pi(y')$, respectively. Then, for each $t \geq 0$, we have

$$\begin{aligned} & |\pi_i(y)(t) - \pi_i(y')(t)| \\ &= \left| \sup_{0 \leq s \leq t} (x_i(s) - (Qy(s))_i)^- - \sup_{0 \leq s \leq t} (x_i(s) - (Qy'(s))_i)^- \right| \\ &\leq \sup_{0 \leq s \leq t} |(Qy(s))_i - (Qy'(s))_i| \end{aligned}$$

where the inequality follows from (3.10). Taking the maximum over i , one has that, for each $t \geq 0$,

$$\begin{aligned} |\pi(y)(t) - \pi(y')(t)| &\leq \max_i \sup_{0 \leq s \leq t} |(Qy(s))_i - (Qy'(s))_i| \\ &= \sup_{0 \leq s \leq t} \max_i |(Qy(s))_i - (Qy'(s))_i| \\ &= \sup_{0 \leq s \leq t} |Qy(s) - Qy'(s)| \\ &\leq \|Q\| \sup_{0 \leq s \leq t} |y(s) - y'(s)|. \end{aligned}$$

Thus, $\|\pi(y) - \pi(y')\|_T \leq \|Q\| \|y - y'\|_T$ or equivalently

$$\rho_T(\pi(y), \pi(y')) \leq \|Q\| \rho_T(y, y'), \quad (3.24)$$

where $\rho_T(\cdot, \cdot)$ is the distance induced by the norm $\|\cdot\|_T$. Hence π is a contraction on $(\mathbb{D}_0^d[0, T], \rho_T)$. Since $(\mathbb{D}_0^d[0, T], \rho_T)$ is a complete metric space, (being a closed subset of the Banach space of bounded functions defined from $[0, T]$ into $\mathbb{R}^d$ and endowed with the uniform norm,) π has a unique fixed point $y \in \mathbb{D}_0^d[0, T]$. Adding a subscript T to the solution y to indicate its dependence on T , we see by uniqueness that for $T' > T$, $y_{T'}$ must agree with y_T on $[0, T]$. Using this fact, we can let $T \rightarrow \infty$ to obtain $y = \lim_{T \rightarrow \infty} y_T$ as the unique solution to (3.21)–(3.22) on the time interval $[0, \infty)$. In fact, we have the following explicit construction for y . For all $t \geq 0$, let

$$\begin{aligned} y^0(t) &= 0 \\ y^1(t) &= \pi(y^0(t)) \equiv \sup_{0 \leq s \leq t} x^-(s) \\ &\vdots \\ y^n(t) &= \pi(y^{n-1}(t)) \equiv \sup_{0 \leq s \leq t} (x(s) - Qy^{n-1}(s))^- \\ &\vdots \end{aligned}$$

The contraction property implies that $\{y^n\}$ is a Cauchy sequence in the complete metric space $(\mathbb{D}_0^d[0, T], \rho_T)$ and it converges to the fixed point y of π . Hence, $\rho_T(y^n, y) \rightarrow 0$ as $n \rightarrow \infty$ for each $T > 0$, i.e., $y^n \rightarrow y$ uniformly on each compact time interval. Finally, z is defined in terms of y by (3.23). $\square$

Exercise 3.3.3. Assume Q is a nonnegative matrix with spectral radius less than one. Prove that there exists a diagonal matrix B with positive diagonal entries such that $\|BQB^{-1}\| < 1$.

By Theorem 3.3.2, when the reflection matrix is a generalized $\mathcal{M}$ -matrix, the reflection map

$$x \in \mathbb{D}_+^d \rightarrow (\phi(x), \psi(x)) = (z, y) \in \mathbb{D}_+^d \times \mathbb{D}_+^d \quad (3.25)$$

is well defined. The next two theorems present the continuity of the map. The first theorem shows that the map is Lipschitz continuous under the uniform norm over finite time intervals.

Theorem 3.3.4. Assume that R is a generalized $\mathcal{M}$ -matrix. There exists a constant $K > 0$ that depends on R such that for $x, x' \in \mathbb{D}_+^d$ and each $T \geq 0$,

$$\|\phi(x) - \phi(x')\|_T \vee \|\psi(x) - \psi(x')\|_T \leq K\|x - x'\|_T, \quad (3.26)$$

where for $a, b \in \mathbb{R}$, $a \vee b = \max(a, b)$.

Proof. Assume that $R = I - Q$ with $\|Q\| < 1$. Let $x, x' \in \mathbb{D}_+^d$, and let (z, y) and (z', y') be the R -regulations for x and x' , respectively. By the proof of Theorem 3.3.2, y and y' are the fixed points defined by the map in (3.20). Thus,

$$y(t) - y'(t) = \sup_{0 \leq s \leq t} (x(s) - Qy(s))^- - \sup_{0 \leq s \leq t} (x'(s) - Qy'(s))^-,$$

where the negative part and the supremum are taken componentwise. Thus, similar to the derivation of (3.24), for any $T > 0$,

$$\|y - y'\|_T \leq \|x - x'\|_T + \|Q\| \|y - y'\|_T,$$

from which we have

$$\|y - y'\|_T \leq \frac{1}{1 - \|Q\|} \|x - x'\|_T. \quad (3.27)$$

For a general reflection matrix R that is a generalized $\mathcal{M}$ -matrix, by the first paragraph of the proof of Theorem 3.3.2, there exist diagonal matrices B_1 and B_2 with positive diagonal entries, and a matrix Q with $\|Q\| < 1$ such that (z, y) is an R -regulation for x if and only if $(B_1 z, B_1 y)$ is an $(I - Q)$ -regulation for $B_2 x$.

Thus,

$$\begin{aligned}
\|y - y'\|_T &\leq K_1 \|B_1 y - B_1 y'\|_T \\
&\leq K_1 \left(\frac{1}{1 - \|Q\|} \right) \|B_2 x - B_2 x'\|_T \\
&\leq K_1 \left(\frac{1}{1 - \|Q\|} \right) K_2 \|x - x'\|_T,
\end{aligned}$$

where K_1 and K_2 are two constants depending on B_1 and B_2 , and the second inequality follows from (3.27). Since

$$\|z - z'\|_T \leq \|x - x'\|_T + \|R\| \|y - y'\|_T,$$

by letting $K = \|R\|K_1K_2/(1 - \|Q\|) + 1$, we have (3.26). $\square$

We end this section by proving the reflection map

$$(\phi, \psi) : x \in \mathbb{D}_+^d \rightarrow (\phi(x), \psi(x)) \in \mathbb{D}_+^{2d} \quad (3.28)$$

is continuous under the Skorokhod J_1 -topology in domain and range. Let Λ be the set of (strictly) increasing continuous functions $\gamma : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $\gamma(0) = 0$ and $\lim_{t \rightarrow \infty} \gamma(t) = \infty$. For $x \in \mathbb{D}^d$ and $\gamma : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, let $x \circ \gamma$ be the composition of x and γ , i.e., $(x \circ \gamma)(t) = x(\gamma(t))$ for $t \geq 0$. Recall that a sequence $\{x^n, n \geq 1\} \subset \mathbb{D}^d$ converges to $x \in \mathbb{D}^d$ in the Skorokhod J_1 -topology, as $n \rightarrow \infty$, if for each $T > 0$ there exists a sequence $\{\gamma^n, n \geq 1\} \subset \Lambda$ such that

$$\|x^n \circ \gamma^n - x\|_T \rightarrow 0 \quad \text{and} \quad \|\gamma^n - e\|_T \rightarrow 0 \quad \text{as } n \rightarrow \infty, \quad (3.29)$$

where $e(t) = t$ for $t \geq 0$. Note that as sets $\mathbb{D}_+^{2d} = \mathbb{D}_+^d \times \mathbb{D}_+^d$. However, as topological spaces, they are different. The latter is endowed with the standard product topology which is weaker than the J_1 -topology on $\mathbb{D}_+^{2d}$. Thus, the continuity of the reflection map in (3.28), when the range is endowed with the Skorokhod J_1 -topology, implies the continuity of the reflection map in (3.25), when the range is endowed with the product topology.

The following lemma shows that when R is a generalized $\mathcal{M}$ -matrix the reflection map commutes with scaling and composition operations. The lemma will be used to prove the reflection map is continuous in the Skorokhod J_1 -topology.

Lemma 3.3.5. *Assume R is a generalized $\mathcal{M}$ -matrix. Let $\alpha > 0$ and $\gamma \in \Lambda$. If $x \in \mathbb{D}_+^d$, then $\alpha(x \circ \gamma) \in \mathbb{D}_+^d$. Furthermore,*

$$\phi(\alpha(x \circ \gamma)) = \alpha\phi(x) \circ \gamma \quad \text{and} \quad \psi(\alpha(x \circ \gamma)) = \alpha\psi(x) \circ \gamma.$$

Proof. Assume that $x \in \mathbb{D}_+^d$. It is clear that $\alpha(x \circ \gamma)$ is right continuous with left limits. Also $\alpha(x \circ \gamma)(0) = \alpha x(0) \geq 0$. Thus, $\alpha(x \circ \gamma) \in \mathbb{D}_+^d$. Let (z, y) be the R -regulation for x , i.e., $z = \phi(x)$ and $y = \psi(x)$. We now show that $(\alpha z \circ \gamma, \alpha y \circ \gamma)$ is an R -regulation for $\alpha x \circ \gamma$. First $\alpha z(\gamma(t)) = \alpha x(\gamma(t)) + R\alpha y(\gamma(t))$ for all $t \geq 0$, since $z(t) = x(t) + Ry(t)$ for all $t \geq 0$. Clearly, $\alpha z(\gamma(t)) \geq 0$ for all $t \geq 0$, since $z(t) \geq 0$ for all $t \geq 0$. For each $i = 1, \dots, d$, $\alpha y_i(\gamma(t))$ is non-decreasing in t , and $\alpha y_i(\gamma(0)) = \alpha y_i(0) = 0$. Finally,

$$\int_{(0, \infty)} \alpha z_i(\gamma(t)) d(\alpha y_i(\gamma(t))) = \alpha^2 \int_{(0, \infty)} z_i(t) dy_i(t) = 0.$$

Therefore, we have proved that $(\alpha z \circ \gamma, \alpha y \circ \gamma)$ is an R -regulation for $\alpha x \circ \gamma$. By the uniqueness of the R -regulation, we have that $\alpha z \circ \gamma = \phi(\alpha x \circ \gamma)$ and $\alpha y \circ \gamma = \psi(\alpha x \circ \gamma)$, thus proving the lemma. $\square$

Theorem 3.3.6. *Assume that R is a generalized $\mathcal{M}$ -matrix. The reflection map $(\phi, \psi) : \mathbb{D}_+^d \rightarrow \mathbb{D}_+^{2d}$ associated with R is continuous when the domain $\mathbb{D}_+^d$ and the range $\mathbb{D}_+^{2d}$ are both endowed with the Skorokhod J_1 -topology.*

Proof. Assume that $\{x^n, n \geq 1\} \subset \mathbb{D}_+^d$ is a sequence that converges to $x \in \mathbb{D}_+^d$ in the Skorokhod J_1 -topology. It suffices to prove that $(\phi(x^n), \psi(x^n))$ converges to $(\phi(x), \psi(x))$ as $n \rightarrow \infty$ in $\mathbb{D}_+^{2d}$ that is endowed with the Skorokhod J_1 -topology. To see this, for each $T > 0$, there exists a sequence $\{\gamma^n\} \subset \Lambda$ such that (3.29) holds. It follows from Lemma 3.3.5 and Theorem 3.3.4 that

$$\begin{aligned} & \|(\phi(x^n), \psi(x^n)) \circ \gamma^n - (\phi(x), \psi(x))\|_T \\ & \leq \|\phi(x^n \circ \gamma^n) - \phi(x)\|_T + \|\psi(x^n \circ \gamma^n) - \psi(x)\|_T \\ & \leq 2K\|x^n \circ \gamma^n - x\|_T. \end{aligned}$$

Thus, $\|(\phi(x^n), \psi(x^n)) \circ \gamma^n - (\phi(x), \psi(x))\|_T \rightarrow 0$ as $n \rightarrow \infty$. Since $\|\gamma^n - e\|_T \rightarrow 0$ as $n \rightarrow \infty$, and $T > 0$ is arbitrary, we have proved that $(\phi(x^n), \psi(x^n))$ converges to $(\phi(x), \psi(x))$ in $\mathbb{D}_+^{2d}$. Thus, the theorem is proved. $\square$

3.4 Existence

For some queueing network, the reflection matrix of the corresponding Skorokhod problem is not a generalized $\mathcal{M}$ -matrix (see Chapter 6). Thus, one cannot invoke Theorem 3.3.2 for the existence of a solution to the Skorokhod problem.

Let $d \geq 1$ be a fixed integer, and R be a $d \times d$ (real) matrix. In this section, we prove that an R -regulation of x exists for each $x \in \mathbb{D}_+^d$ if and only if R is a completely- $\mathcal{S}$ matrix, a notion we now define. Recall that for each nonempty subset $\mathcal{I} \subseteq \{1, \dots, d\}$, the principal submatrix $R^{\mathcal{I}}$ of R associated with $\mathcal{I}$ is the matrix obtained by deleting all rows and columns of R with indices $i \notin \mathcal{I}$.

Definition 3.4.1 (Completely- $\mathcal{S}$). A $d \times d$ matrix R is said to be an $\mathcal{S}$ matrix if there exists a $v \in \mathbb{R}_+^d$ such that $Rv > 0$. The matrix R is said to be *completely- $\mathcal{S}$* if each principal submatrix is an $\mathcal{S}$ matrix.

When a reflection matrix R is completely- $\mathcal{S}$, we first prove that an R -regulation of a simple path exists.

Definition 3.4.2. A path $x \in \mathbb{D}^d$ is said to be a *simple path* if x has the following form:

$$x(t) = \sum_{i=0}^{n-1} x(t_i) \mathbf{1}_{[t_i, t_{i+1})}(t) + x(t_n) \mathbf{1}_{[t_n, \infty)}(t), \quad t \geq 0, \quad (3.30)$$

where n is some nonnegative integer, and $0 = t_0 < t_1 < \dots < t_n$.

To prove an R -regulation of a simple path exists, we introduce discretized version of the Skorokhod problem.

Definition 3.4.3 (Linear Complementarity Problem). Let d be a positive integer, and R be a $d \times d$ matrix. Fix $u \in \mathbb{R}^d$. A pair $(v, w) \in \mathbb{R}^d \times \mathbb{R}_+^d$ is a solution of the Linear Complementarity Problem (LCP) for u associated with R if

$$(i.) \quad w = u + Rv,$$

$$(ii.) \quad w'v = 0,$$

where, as before, w' denotes the transpose of w .

Lemma 3.4.4. Assume that a $d \times d$ matrix R is completely- $\mathcal{S}$. Then for each point $u \in \mathbb{R}^d$, there exists at least one solution to the LCP for u associated with R .

We leave a proof of Lemma 3.4.4 to the end of this section.

We now state and prove the following lemma.

Lemma 3.4.5. Let d be a positive integer, and R be a $d \times d$ completely- $\mathcal{S}$ matrix. For each simple path x , an R -regulation of x exists.

Proof. Assume that R is completely- $\mathcal{S}$, and $x \in \mathbb{D}_+^d$ is a simple path of the form in (3.30), where $n \geq 0$ is an integer and $t_0 = 0 < t_1 < \dots < t_n$. Since $x(t) = x(0) \geq 0$ for $t_0 \leq t < t_1$, by defining $y(t) = 0$ for $t_0 \leq t < t_1$ and $z(t) = x(t)$ for $t_0 \leq t < t_1$, one can easily check that (z, y) is an R -regulation of x , restricted to the time interval $[t_0, t_1]$. To extend the R -regulation to $[t_0, t_2]$, let $u_1 = x(t_1)$, and (w_1, v_1) is an LCP of u . Let $z(t) = w_1$ and $y(t) = v_1$ for $t_1 \leq t < t_2$. It follows that for $t_1 \leq t < t_2$, $z(t) = w_1 = u_1 + Rv_1 = x(t) + Ry(t)$. Suppose that an R -regulation (z, y) for x has been defined on $[0, t_i]$ for $i < n$. We now construct an R -regulation for x on $[0, t_{i+1}]$. Let $u_i = z(t_{i-1}) + x(t_i) - x(t_{i-1})$, and (w_i, v_i) be an LCP for u_i . Let $z(t) = w_i$ and $y(t) = y(t_{i-1}) + v_i$. One can check that for

$$\begin{aligned}
 z(t) &= w_i = u_i + Rv_i \\
 &= z(t_{i-1}) + x(t_i) - x(t_{i-1}) + Rv_i \\
 &= x(t_{i-1}) + Ry(t_{i-1}) + x(t_i) - x(t_{i-1}) + Rv_i \\
 &= x(t_i) + R(y(t_{i-1}) + v_i) \\
 &= x(t) + Ry(t).
 \end{aligned}$$

□

3.5 Oscillation Inequalities

In this section, we first define a family of perturbed Skorokhod problems with perturbation parameter $\delta \geq 0$. We then prove oscillation inequalities for solutions of a perturbed Skorokhod problem. When $\delta = 0$, the oscillation inequalities are the key tool used in Section 3.4 to prove the existence of a solution of a Skorokhod problem. These inequalities are also the key to the proof of the existence of the tightness result claimed in Theorem x in Chapter y. The oscillation inequality given below is a generalization of one given by Bernard and El Kharroubi [Bernard and El Kharroubi \(1991\)](#) for the case where $x(\cdot)$ is continuous and $\delta = 0$.

In this section, for any $0 \leq t_1 < t_2 < \infty$, $\mathbb{D}([t_1, t_2], \mathbb{R}^d)$ denotes the set of functions $z : [t_1, t_2] \rightarrow \mathbb{R}^d$ that are right continuous on $[t_1, t_2]$ and have finite left limits on $(t_1, t_2]$.

Theorem 3.5.1. *Assume that R is completely- $\mathcal{S}$. Suppose that $\delta \geq 0$, $0 \leq t_1 < t_2 < \infty$ and $z, x, y \in \mathbb{D}([t_1, t_2], \mathbb{R}^d)$ are such that*

- (i) $z(t) = x(t) + Ry(t)$ for all $t \in [t_1, t_2]$,