

Skorohod Repn Thm

(followed proof in Billingsley, Converg. of Prob. Meas.) ①

$(M, \mathcal{M})$ separable m-space.

$(\pi^n)_{n \geq 1}, \pi$ prob. measures on M s.t. $\pi^n \xrightarrow{n \rightarrow \infty} \pi$

Then $\exists (\Omega, \mathcal{F}, P)$ supporting random entities $(X_n)_{n \geq 1}$ and X such that

$$\bullet \mathcal{L}(X_n) = \pi^n \quad \forall n, \quad \mathcal{L}(X) = \pi$$

$$\bullet X_n \rightarrow X \text{ P-a.s. as } n \rightarrow \infty;$$

i.e. for P-a.e. ω , $\underbrace{X_n(\omega) \rightarrow X(\omega)}_{\text{in } M\text{'s metric.}} \text{ as } n \rightarrow \infty.$

Recall Portmanteau Thm

$$\pi^n \Rightarrow \pi \equiv \int_M f d\pi^n \rightarrow \int_M f d\pi \quad \forall f \in C_b(M)$$

$$\text{Portmanteau} \Leftrightarrow \limsup_{n \rightarrow \infty} \pi_n^*(F) \leq \pi(F) \quad \forall F \text{ closed}$$

$$\liminf_{n \rightarrow \infty} \pi^n(G) \geq \pi(G) \quad \forall G \text{ open}$$

Suppose $A \in \mathcal{M}$ with $\pi(\partial A) = 0$, $\pi^n \Rightarrow \pi$.

$$\text{Then } \pi(\bar{A}) \geq \limsup_{n \rightarrow \infty} \pi^n(\bar{A})$$

$$\geq \limsup_{n \rightarrow \infty} \pi^n(A)$$

$$\geq \liminf_{n \rightarrow \infty} \pi^n(A)$$

$$\geq \liminf_{n \rightarrow \infty} \pi^n(A^\circ)$$

$$\geq \pi(A^\circ)$$

$$\text{But } \pi(\bar{A}) = \pi(A^\circ) + \pi(\partial A) = \pi(A^\circ) \quad \text{since } \pi(\partial A) = 0$$

$\therefore$ all " $\geq$ " are " $=$ "

$$\Rightarrow \limsup_{n \rightarrow \infty} \pi^n(A) = \liminf_{n \rightarrow \infty} \pi^n(A) \equiv \lim_{n \rightarrow \infty} \pi^n(A)$$

$$\therefore \lim_{n \rightarrow \infty} \pi^n(A) = \pi(A)$$

$\therefore \pi^n \Rightarrow \pi$ implies $\pi^n(A) \rightarrow \pi(A) \quad \forall$
 π -continuity sets A .

Claim 1 : $\forall \varepsilon > 0 \exists$ partition of M $(B_0, B_1, \dots, B_k)$ (2)

with each $B_i \in \mathcal{M}$ satisfying

$$i) \pi(B_0) < \varepsilon$$

$$ii) \pi(\partial B_i) = 0 \quad \forall i = 0, \dots, k$$

$$iii) \text{diam}(B_i) < \varepsilon \quad \forall i = 1, \dots, k.$$

Define open cover of M by:

For each $x \in M$, choose $\varepsilon/4 < r_x < \varepsilon/2$

s.t. $\pi(\partial B(x, r_x)) = 0$.

M separable $\Rightarrow \exists$ countable $\{x_1, x_2, \dots\} \subset M$

$$\text{s.t.} \quad M \subset \bigcup_{i=1}^{\infty} \underbrace{B(x_i, r_{x_i})}_{:= A_i}$$

$$\text{Note} \quad \lim_{k \rightarrow \infty} \pi\left(\bigcup_{i=1}^k A_i\right) = \pi\left(\bigcup_{i=1}^{\infty} A_i\right) \geq \pi(M) = 1$$

$$\therefore \exists k \text{ s.t. } \pi\left(\bigcup_{i=1}^k A_i\right) > 1 - \varepsilon.$$

Define

$$B_1 = A_1$$

$$B_2 = A_1^c \cap A_2$$

$$\vdots$$

$$B_k = A_1^c \cap \dots \cap A_{k-1}^c \cap A_k$$

$$B_0 = \left(\bigcup_{i=1}^k A_i\right)^c$$

i) $\checkmark$

ii) $B_i \subset A_i$ for $i=1, \dots \checkmark$

$$ii) \quad \partial B_i \subset \bigcup_{j=1}^i \partial A_j$$

$\checkmark$

$\pi(\uparrow) = 0$ by construction of r_x 's.

(see next pg. for details on (ii)).

ii) Want to show: $\pi(\partial B_i) = \emptyset$ for $i=0, \dots, k$

(2'12)

(Assume $1 \leq i \leq k$; argument for $i=0$ is similar).

Note that by construction of the τ_x ,

$$\pi\left(\bigcup_{i=1}^k \partial A_i\right) = \emptyset$$

So it will be sufficient to show $\partial B_i \subset \bigcup_{j=1}^i \partial A_j$:

Let $x \in \partial B_i$. Then

(a) $\exists \{x_n\} \subset B_i$ s.t. $x_n \rightarrow x$

(b) $\exists \{y_n\} \subset B_i^c$ s.t. $y_n \rightarrow x$

Note $B_i^c = A_1 \cup \dots \cup A_{i-1} \cup A_i^c$

$\therefore \{y_n\} \subset (A_1 \cup \dots \cup A_{i-1} \cup A_i^c)$

$\therefore \exists$ infinite subsequence $\{y_{n_k}\} \subset A_j$ $\forall 1 \leq j \leq i-1$

OR $\{y_{n_k}\} \subset A_i^c$.

- If $\{y_{n_k}\} \subset A_j$ $\forall 1 \leq j \leq i-1$, ~~then we conclude~~ and also by (a) $\{x_n\} \subset B_i \subset A_j$ is such that $x_n \rightarrow x$, then we conclude $x \in A_j$.

- If $\{y_{n_k}\} \subset A_i^c$, and also by (a) $\{x_n\} \subset B_i \subset A_i$ is such that $x_n \rightarrow x$, then we conclude $x \in A_i$.

$\therefore x \in \bigcup_{j=1}^i \partial A_j$, as desired.

Now construct a whole seq of partitions:

(3)

$\forall \varepsilon_m := 1/2^m \exists$ partition of M ($B_0^m, B_1^m, \dots, B_{k_m}^m$)

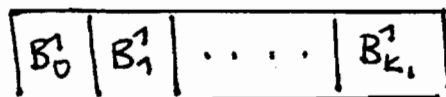
with each $B_i^m \in \mathcal{M}$ sat.

$$i) \pi(B_0^m) < \varepsilon_m$$

$$ii) \pi(B_i^m) = 0 \quad \forall i = 0, \dots, k_m$$

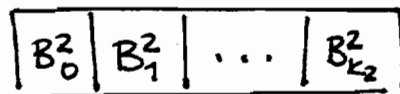
$$iii) \text{diam}(B_i^m) < \varepsilon_m \quad \forall i = 1, \dots, k_m$$

$$m=1 \quad \varepsilon_1 = 1/2, \quad k_1$$



stick here all the B_i^1 s.t. $\pi(B_i^1) = 0$

$$m=2 \quad \varepsilon_2 = (1/2)^2, \quad k_2$$



⋮

By (ii), every B_i^m is a π -continuity set, so

$$\pi^n(B_i^m) \longrightarrow \pi(B_i^m) \quad \forall m$$

$$\forall 0 \leq i \leq k_m$$

$\therefore$ For each m ,

$$\pi^n(B_0^m) \longrightarrow \pi(B_0^m) \quad \text{so } \exists n_0 \text{ s.t. } n \geq n_0 \Rightarrow \pi^n(B_0^m) \geq (1 - \varepsilon_m) \pi(B_0^m)$$

$$\pi^n(B_1^m) \longrightarrow \pi(B_1^m) \quad \text{so } \exists n_1 \text{ s.t. } n \geq n_1 \Rightarrow \pi^n(B_1^m) \geq (1 - \varepsilon_m) \pi(B_1^m)$$

⋮

$$\pi^n(B_{k_m}^m) \longrightarrow \pi(B_{k_m}^m) \quad \text{so } \exists n_{k_m} \dots$$

$$\text{Let } N_m = \max \{n_0, n_1, \dots, n_{k_m}\}$$

wlog, assume $N_1 < N_2 < N_3 < \dots$

(by redefining $\tilde{N}_m$ as $\max \{N_m, N_{m-1} + 1\}$)

$$\left[\text{Then } \forall n \geq N_m, \pi^n(B_i^m) \geq (1 - \varepsilon_m) \pi(B_i^m) \quad \forall i = 0, \dots, k_m \right] (*)$$

NOW find $(\mathcal{Q}, \mathcal{F}, \mathcal{P})$ supporting random entities

(4)

$$X \text{ s.t. } \mathcal{Q}(X) = \pi$$

$$\{Y_n\}_{n \geq 1} \text{ s.t. } \mathcal{Q}(Y_n) = \pi^n \quad \forall n$$

$$(*_2) \quad \{Y_n^i\}_{n \geq 1}^{i \geq 0} \text{ s.t. } \mathcal{Q}(Y_n^i) = \pi^n(\cdot | B_i^m) \text{ where } N_m \leq n < N_{m+1}$$

$$\{Z_n\}_{n \geq 1} \text{ s.t. } \mathcal{Q}(Z_n) = \mu^n \text{ (defined below)}$$

taking values in $\mathbb{M}$, and a random variable

$$\xi: \Omega \rightarrow \mathbb{R} \text{ s.t. } \xi \sim \mathcal{U}([0, 1])$$

so that all of the above are independent.

$(*_2)$: If, for some m , there exist $N_m \leq n < N_{m+1}$ and $0 \leq i \leq k_m$ such that $\pi^n(B_i^m) = 0$, we set $\pi^n(A | B_i^m) = 0 \quad \forall A$.

Define μ^n , a probability measure on $(\mathbb{M}, \mathcal{M})$, by

$$\mu^n(A) = \frac{1}{\varepsilon_m} \sum_{i=0}^{k_m} \pi^n(A | B_i^m) [\pi^n(B_i^m) - (1 - \varepsilon_m) \pi(B_i^m)]$$

where m is such that $N_m \leq n < N_{m+1}$.

Remark: by $(*)$ (p.3), if $\pi^n(B_i^m) = 0$ for some i , then $\pi(B_i^m) = 0$.

$$\circ \circ \quad \varepsilon_m \mu^n(\mathbb{M}) = \sum_{i=0}^{k_m} \pi^n(\mathbb{M} | B_i^m) [\pi^n(B_i^m) - (1 - \varepsilon_m) \pi(B_i^m)]$$

$$= \sum_I \pi^n(\mathbb{M} | B_i^m) [\pi^n(B_i^m) - (1 - \varepsilon_m) \pi(B_i^m)]$$

$$\text{where } I = \{0 \leq i \leq k_m \mid \pi^n(B_i^m) > 0\}$$

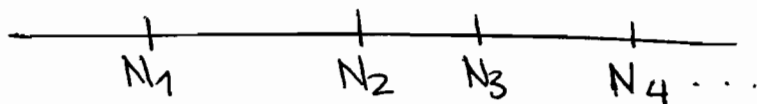
$$= \underbrace{\sum_I \pi^n(B_i^m)}_{=1} - \sum_I \underbrace{\pi^n(\mathbb{M} | B_i^m)}_{=1 \text{ since } \pi^n(B_i^m) \neq 0} \pi(B_i^m) + \varepsilon_m \sum_I \underbrace{\pi^n(\mathbb{M} | B_i^m)}_{=1} \pi(B_i^m)$$

$$= 1 - 1 + \varepsilon_m \text{ by Remark}$$

$$= \varepsilon_m, \text{ as desired.}$$

Define $\{X_n\}_{n \geq 1}$ as follows:

(5)



• If $n < N_1$, set $X_n = Y_n$

• If $N_m \leq n < N_{m+1}$, set

$$X_n = \left(\sum_{i=0}^{k_m} 1_{[\xi \leq 1 - \varepsilon_m, X \in B_i^m]} Y_n^i \right) + 1_{[\xi > 1 - \varepsilon_m]} Z_n$$

Note: For each n , entire sum has exactly one nonzero summand.

We have $\mathcal{L}(X_n) = \Pi^n$ since, for $A \in \mathcal{M}$,

$$P(X_n \in A) = \left(\sum_{i=0}^{k_m} P[\xi \leq 1 - \varepsilon_m \text{ and } X \in B_i^m] \cdot P(Y_n^i \in A) \right) + P[\xi > 1 - \varepsilon_m] \cdot P(Z_n \in A)$$

$$= \left(\sum_{i=0}^{k_m} (1 - \varepsilon_m) \Pi(B_i^m) \Pi^n(A | B_i^m) \right) + \varepsilon_m \mu^n(A)$$

$$= \sum_{i=0}^{k_m} (1 - \varepsilon_m) \Pi(B_i^m) \Pi^n(A | B_i^m) + \sum_{i=0}^{k_m} \Pi^n(A | B_i^m) [\Pi^n(B_i^m) - (1 - \varepsilon_m) \Pi(B_i^m)]$$

$$= \sum_{i=0}^{k_m} \Pi^n(A | B_i^m) \Pi^n(B_i^m)$$

$$= \sum_{i=0}^{k_m} \Pi^n(A \cap B_i^m)$$

$$= \Pi^n(A)$$

which implies that $\mathcal{L}(X_n) = \Pi^n$.

For each m , we define

$$E_m := \{\omega \in \Omega \mid X(\omega) \notin B_0^m \text{ and } \varepsilon(\omega) \leq 1 - \varepsilon_m\}$$

$$E := \liminf_{m \rightarrow \infty} E_m = \bigcup_{m=1}^{\infty} \bigcap_{k=m}^{\infty} E_k.$$

For fixed n , $\exists ! m$ s.t. $N_m \leq n < N_{m+1}$.

Claim: On E_m , $P(X_n \in B_j^m \mid X \in B_j^m) = 1$

Pf of claim:

$$\begin{aligned} & P(X_n \in B_j^m \mid X 1_{E_m} \in B_j^m) \\ &= \sum_{i=0}^{k_m} P(1_{[\varepsilon \leq 1 - \varepsilon_m, X \in B_i^m]} Y_n^i \in B_j^m \mid X \in B_j^m) \\ & \quad \text{only } = 1 \text{ when } i=j \end{aligned}$$

$$= P(Y_n^j \in B_j^m \mid X \in B_j^m)$$

$$\stackrel{(II)}{=} P(Y_n^j \in B_j^m)$$

$$\lfloor = \pi^n(B_j^m \mid B_j^m) = 1, \text{ as desired.}$$

Fix $\varepsilon > 0$. $\exists K$ s.t. $\varepsilon_k < \varepsilon$.

If $\omega \in E$, $\exists L$ s.t. $\omega \in E_l \forall l \geq L$.

Take $N = \max\{K, L\}$.

Then $n \geq N \Rightarrow |X_n(\omega) - X(\omega)| < \varepsilon$ with prob. 1.

$\therefore X_n \xrightarrow{\text{a.s.}} X$ on E .

Recall the FIRST BOREL-Cantelli lemma:

(7)

$$\sum_n P(A_n) < \infty \implies P(\limsup_n A_n) = 0.$$

Claim: $P(E) = 1$.

pf. of claim:

$$\begin{aligned} P(E_m^c) &= P(\{\omega \mid X(\omega) \in B_0^m \text{ OR } \xi(\omega) > 1 - \varepsilon_m\}) \\ &< 2\varepsilon_m \\ &= 2\left(\frac{1}{2}\right)^m \end{aligned}$$

$$\therefore \sum_m P(E_m^c) < \infty$$

$$\stackrel{\text{B-C}}{\implies} P(\limsup_m E_m^c) = 0$$

$$\implies P(\liminf_m E_m)^c = 0$$

$$\implies P(\liminf_m E_m) = 1; \text{ i.e. } P(E) = 1 \quad \checkmark$$

L

Without changing their laws, we can redefine X_n, X to be identically zero on E^c , which allows us to conclude that

$$X_n \longrightarrow X \text{ P-a.s. as } n \rightarrow \infty.$$

