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Consider $\{X^n\}_{n=1}^\infty$ a sequence of d -dim stochastic processes

Let π^n be the law of X^n on (D^n, D^n)

$\{\pi_n\}_{n=1}^\infty$ is tight iff $\{X_n\}_{n=1}^\infty$ is tight.

Thm: The sequence $\{X_n\}_{n=1}^\infty$ is tight iff for each $T > 0$ and $\varepsilon > 0$

$$(i) \lim_{K \rightarrow \infty} \overline{\lim}_{n \rightarrow \infty} P(\|X^n\|_T \geq K) = 0$$

$$(ii) \lim_{\delta \rightarrow 0} \overline{\lim}_{n \rightarrow \infty} P(W(X^n, \delta, T) \geq \varepsilon) = 0$$

Usual modulus of continuity

$$W(X, \delta, T) = \sup_{\substack{u, v \in [0, T] \\ |u-v| \leq \delta}} |X(u) - X(v)|, \quad X \in \mathbb{D}^d$$

Definition: $\{X_n\}_{n=1}^\infty$ is C-tight if

(i) it is tight

(ii) any weak limit ^{point} of $\{X_n\}$ has continuous paths almost surely.

Thm: The sequence $\{X_n\}_{n=1}^\infty$ is C-tight iff for each $T > 0$ and $\varepsilon > 0$

$$(i) \lim_{K \rightarrow \infty} \overline{\lim}_{n \rightarrow \infty} P(\|X^n\|_T \geq K) = 0$$

$$(ii) \lim_{\delta \rightarrow 0} \overline{\lim}_{n \rightarrow \infty} P(W(X^n, \delta, T) \geq \varepsilon) = 0$$

A weak ^{point} limit of $(X_n)_{n \in \mathbb{N}}$ is a stochastic process X such that $X^{n_k} \Rightarrow X$ as $k \rightarrow \infty$ for some subsequence $\{n_k\}$ of $\{n\}$.

Random Time Change Lemma

$X \in \mathbb{D}^d$. $\varphi \in \mathcal{D}_0 = \{\psi: [0, \infty) \rightarrow [0, \infty), \psi \text{ is non-decreasing, } \psi \text{ is r.c.l.l.}\}$

Exercise: Check $X \circ \varphi \in \mathbb{D}^d$

Facts:

(1) For $t \in [0, \infty)$, $x \mapsto X(t)$, for $X \in \mathbb{D}^d$ is continuous at x if X is continuous at t .

PF: Suppose $X \in \mathbb{D}^d$ and X is continuous at t . Suppose $X_n \rightarrow X$ in $\mathbb{D}^d$. Let $T > t$. $\exists$ sequence of time changes $\{\gamma_n\}_{n=1}^\infty \subset \Gamma$.

(+) $\|\gamma_n(\cdot) - e(\cdot)\|_T \rightarrow 0$ as $n \rightarrow \infty$ where $e(t) = t \quad \forall t \geq 0$

(*) $\|X_n(\cdot) - X(\gamma_n(\cdot))\|_T \rightarrow 0$ as $n \rightarrow \infty$.

$$|X_n(t) - X(t)| \leq \underbrace{|X_n(t) - X(\gamma_n(t))|}_{\substack{\rightarrow 0 \text{ as } n \rightarrow \infty \\ \text{by (*)}}} + \underbrace{|X(\gamma_n(t)) - X(t)|}_{\substack{\rightarrow 0 \text{ as } n \rightarrow \infty \text{ since} \\ X \text{ is continuous at } t \\ \text{by (+)}}}$$

Since X is discontinuous at at most countably many points,

$X_n(t) \rightarrow X(t)$ as $n \rightarrow \infty$ for all but countably many t .

Furthermore, if X is continuous at $t_1, \dots, t_m$, then

$$(X_n(t_1), \dots, X_n(t_m)) \rightarrow (X(t_1), \dots, X(t_m)) \text{ as } n \rightarrow \infty$$

(2). For X a d-dimensional process,

$$J_t = \{\omega: X_t(\omega) \neq X_{t-}(\omega)\} \text{ for } t \geq 0$$

$$P(J_t) = 0 \text{ for all but countably many } t.$$

(3). Suppose $X^n \Rightarrow X$ as $n \rightarrow \infty$

By Skorokhod Representation Thm, there are processes $\{\tilde{X}^n\}_{n=1}^\infty, \tilde{X}$ all defined on the same probability space, such that

$$X^n \stackrel{d}{=} \tilde{X}^n \quad \forall n, \quad X \stackrel{d}{=} \tilde{X}, \quad \tilde{X}^n \rightarrow \tilde{X} \text{ a.s. as } n \rightarrow \infty \quad (**)$$

There is a countable set $B \subset [0, \infty)$ such that for each $t \in [0, \infty) \setminus B$,

$$P(\tilde{X} \text{ is continuous at } t) = 1.$$

Hence for any $t_1, t_2, \dots, t_m \in [0, \infty) \setminus B$, for a.e. ω , $\tilde{X}$ is continuous

at $t_1, \dots, t_m$.

So by (**), and (1), $(\tilde{X}^n(t_1), \dots, \tilde{X}^n(t_m)) \rightarrow (\tilde{X}(t_1), \dots, \tilde{X}(t_m))$

a.s. as $n \rightarrow \infty$

Hence, $(X^n(t_1), \dots, X^n(t_m)) \Rightarrow (X(t_1), \dots, X(t_m))$ as $n \rightarrow \infty$.

Summary: $X^n \Rightarrow X$ as $n \rightarrow \infty$ implies convergence of the finite dimensional distributions of X^n to those of X at all but countably many times.

~~Subsets~~. $[X(t_1), \dots, X(t_m)]$ for finite $t_1, \dots, t_m$ subsets of $(0, \infty)$ are the finite dimensional distribution (f.d.d.) of X .

14). If $\{X^n\}_{n=1}^\infty$ is tight and there is a dense set of times

$\mathcal{T} \subset [0, \infty)$ such that $(X^n(t_1), \dots, X^n(t_m)) \Rightarrow (X(t_1), \dots, X(t_m))$

as $n \rightarrow \infty$ for all $t_1, \dots, t_m \in \mathcal{T}$. then $X^n \Rightarrow X$ as $n \rightarrow \infty$

Donsker's Invariance Principles

Recall if $\{U_i\}_{i=1}^\infty$ is an iid. sequence of r.v.'s with finite mean

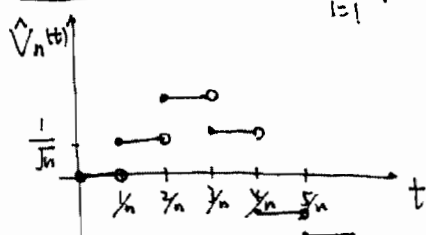
μ and Variance $\sigma^2 > 0$

Central Limit Thm

$$\frac{\sum_{i=1}^n V_i - n\mu}{\sigma\sqrt{n}} \xrightarrow{n \rightarrow \infty} \mathcal{N} \stackrel{d}{=} N(0,1). \quad \text{normal r.v. Mean 0, Variance 1.}$$

$$\frac{\sum_{i=1}^n V_i}{n} \rightarrow \mu \text{ in probability as } n \rightarrow \infty. \quad \text{Weak law of large Num.}$$

Random Walk. $\sum_{i=1}^n (V_i - \mu)$



$$\hat{V}^n(t) = \frac{\sum_{i=1}^{\lfloor nt \rfloor} (V_i - \mu)}{\sigma\sqrt{n}} \quad \lfloor \cdot \rfloor = \text{integer part}$$

Donsker's Thm (Functional Central Limit Thm)

$\{\hat{V}^n(t)\}_{n=1}^{\infty}$ converges in distribution to W , a standard one-dimensional Brownian Motion. (cts paths, independent increments, $W(t) - W(s) \sim N(0, t-s)$, $W(0) = 0$)

Renewal Process ^{Central} ~~General~~ Limit Thm

Let $\{u_i\}_{i=1}^{\infty}$ be i.i.d. (strictly) positive r.v.s with mean $\lambda^{-1} \in (0, \infty)$, and Variance $\sigma^2 \in (0, \infty)$

$$N(t) = \sup \{k: u_1 + u_2 + \dots + u_k \leq t\}$$

renewal process.

(poisson process - exponential u_i 's)

$$\hat{N}^n(t) = \frac{N(nt) - \lambda nt}{\lambda^{3/2} \sigma \sqrt{n}} \quad t \geq 0$$

Then $\hat{N}^n \Rightarrow W$ as $n \rightarrow \infty$ (Wiener - W)

