

Lecture 15

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Last Time: Single Class Queueing Networks

$$\bar{Q}^r \Rightarrow \bar{Q} \quad \text{assuming } \bar{Q}^r(0) \Rightarrow q_0 \text{ as } r \rightarrow \infty$$

where

Fluid Model Equations

$$\begin{cases} \bar{Q}(t) = q_0 + \alpha t - \bar{D}(t) + P' \bar{D}(t) \\ \bar{D}(t) = \text{diag}(\mu) \bar{T}(t) \\ \bar{I}(t) = \underline{1} t - \bar{T}(t), \end{cases} \quad \bar{I}_i \text{ is continuous, non-decreasing \& can only increase when } \bar{Q}_i \text{ is zero, } \forall i$$

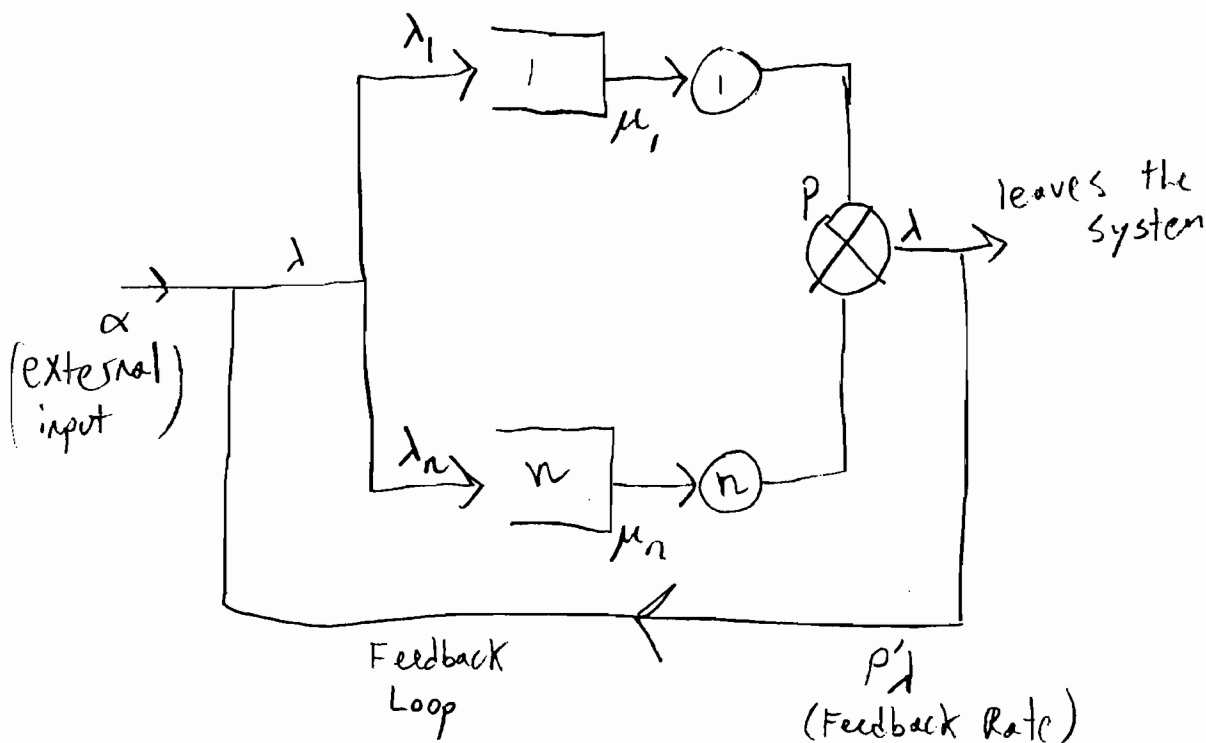
(Note: $\underline{1}$ is the vector of all ones)

Note: $\bar{T}$ is uniformly Lipschitz continuous

Substituting these equations into one another we get

$$\bar{Q}(t) = q_0 + \alpha t - (\underline{I} - P') \mu t + \underbrace{(\underline{I} - P') \text{diag}(\mu)}_R \bar{I}(t)$$

Fluid Schematic



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Recall: Dai's Theorem (1995)

Under Mld assumptions, "If a fluid model is stable then the queueing network is stable."

(see the previous lecture for a discussion of fluid model & queueing network stability)

Traffic Equations

$$\lambda = \alpha + P' \lambda \quad (\text{see schematic})$$

λ = expected average arrival rate

$$(P')^n \rightarrow 0 \text{ as } n \rightarrow \infty$$

Solving for λ we get $(I - P')^{-1} \alpha = \lambda$ since

$$(I - P')^{-1} = I + P' + (P')^2 + \dots$$

The fluid model is stable iff $\lambda < \mu$.

Define the traffic intensity parameter $\rho_i = \frac{\lambda_i}{\mu_i}$.

Proof that $\lambda < \mu$ implies fluid model stability

Linear Lyapunov Function

$$F(t) = \underline{1}' (I - P')^{-1} \bar{Q}(t)$$

$$(\mathbf{I} - \mathbf{P}')^{-1} \bar{\mathbf{Q}}(t) = \bar{\mathbf{Q}}(t) + \mathbf{P}' \bar{\mathbf{Q}}(t) + (\mathbf{P}')^2 \bar{\mathbf{Q}}(t) + \dots$$

= expected # of jobs that need to be served eventually based on jobs currently in the system

$L(\bar{\mathbf{Q}}) = \mathbf{1}'(\mathbf{I} - \mathbf{P}')\bar{\mathbf{Q}}$ has the following properties

1.) Has a minimum of zero at $\bar{\mathbf{Q}} = 0$

2.) We will show
$$\begin{cases} f'(t) \leq -\varepsilon < 0 & \text{for a.e. } t < t_0 = \inf\{s \geq 0 : \bar{\mathbf{Q}}(s) = 0\} \\ f'(t) = 0 & \text{for a.e. } t \geq t_0 \end{cases}$$

Notes on regularity of fluid model solutions:

Since $\bar{\mathbf{T}}$ is uniformly Lipschitz continuous, it is absolutely continuous and so it is differentiable a.e. and can be recovered by integration from that a.e. defined derivative. (plus an initial condition).

Since $\bar{\mathbf{Q}}$ & $\bar{\mathbf{D}}$ can be written in terms of $\bar{\mathbf{T}}$, they also have this property.

Call $t > 0$ a regular point for $\bar{\mathbf{T}}$ if $\bar{\mathbf{T}}$ is differentiable at t .

Consider a regular point $\mathbf{t}$ for $\bar{\mathbf{T}}$ where $\bar{\mathbf{Q}}_i(t) = 0$

Approximating $\dot{\bar{Q}}_i(t)$ from the left we get

$$\dot{\bar{Q}}_i(t) \approx \frac{Q_i(t) - Q_i(t-h)}{h} \leq 0$$

From the right we get

$$\dot{\bar{Q}}_i^+(t) \approx \frac{Q_i(t+h) - Q_i(t)}{h} \geq 0$$

So $\dot{\bar{Q}}_i(t) = 0$ if it exists at t .

$$F(t) = \underline{1}'(\underline{I} - P')^{-1} \bar{Q}(t) = \underline{1}'((\underline{I} - P')^{-1} q_0 + (\underline{I} - P')^{-1} \alpha t - \mu t + \text{diag}(\mu) \bar{I}(t))$$

Let t be a regular point for $\bar{F}$.

$$\dot{F}(t) = \underline{1}'((\lambda - \mu) + \text{diag}(\mu) \bar{I}(t))$$

Let $B(t) = \{i : \bar{Q}_i(t) = 0\}$, $B = B(t)$, $B^c = \{1, \dots, n\} \setminus B$

$$\dot{F}(t) = \sum_{i \in B^c} (\lambda_i - \mu_i) + \sum_{i \in B} (\lambda_i - \mu_i + \mu_i \bar{I}_i(t))$$

Since t is regular, for any $i \in B$,

$$0 = \dot{\bar{Q}}_i(t) = \alpha_i - \dot{D}_i(t) + \sum_j P_{ji}' \dot{D}_j(t)$$

$$\bar{Q}_B = (\bar{Q}_i : i \in B)$$

$$\begin{matrix} B \{ & \begin{matrix} \bullet & \bullet \\ P'_{BB} & P'_{BB^c} \end{matrix} \\ B^c \{ & \begin{matrix} & \\ & \end{matrix} \end{matrix} \quad \begin{matrix} B^c \\ \end{matrix}$$

P'

$$\alpha_B = (\alpha_i : i \in B)$$

$$\therefore \dot{Q}_B(t) = \alpha_B - \dot{D}_B(t) + P'_{BB} \dot{D}_B(t) + P'_{BB^c} \dot{D}_{B^c}(t)$$

Since P' has spectral radius less than one,
 P'_{BB} also does so $I_{BB} - P'_{BB}$ is invertible.

$$\dot{D}_B(t) = (I_{BB} - P'_{BB})^{-1} \alpha_B + (I_{BB} - P'_{BB})^{-1} P'_{BB^c} \underbrace{\dot{D}_{B^c}(t)}_{\mu_{B^c}}$$

$$\text{For } i \in B^c, \dot{Q}_i(t) > 0, \dot{I}_i(t) = 0, \dot{T}_i(t) = 1$$

$$\dot{D}_i(t) = \mu_i$$

$$\dot{D}_B(t) \geq (I_B - P'_{BB})^{-1} \alpha_B + (I_{BB} - P'_{BB})^{-1} P'_{BB^c} \lambda_{B^c} = \lambda_B$$

$$\text{Since } \lambda = \alpha + P'\lambda$$

$$\lambda_B = \alpha_B + P'_{BB} \lambda_B + P'_{BB^c} \lambda_{B^c}$$

$$\Rightarrow \lambda_B = (I - P'_{BB})^{-1} \alpha_B + (I - P'_{BB})^{-1} P'_{BB^c} \lambda_{B^c}$$

$$\text{and since } \dot{D}_{B^c}(t) = \mu_{B^c} < \lambda_{B^c}$$

$$\lambda_B \leq \dot{D}_B(t) = \text{diag}(\mu_B) \dot{T}_B(t)$$

$$\dot{I}(t) = 1_B - \dot{T}(t)$$

$$\dot{I}_i(t) \leq 1 = \lambda_i / \mu_i, \quad i \in B$$

Substituting in $\dot{F}(t)$,

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$$\dot{F}(t) \leq \sum_{i \in B^c} (\lambda_i - \mu_i) + \underbrace{\sum_{i \in B} (\lambda_i - \mu_i + \mu_i - \lambda_i)}_{=0}$$

Let $\varepsilon = \min_{i=1, \dots, n} (\mu_i - \lambda_i)$

$$\dot{F}(t) \leq -\varepsilon \quad \forall t < t_0 = \inf \{s \geq 0 : \bar{Q}(s) = 0\}$$

Also $\dot{F}(t) = 0 \quad \forall t \geq t_0$ if t is regular.

It follows that $F(t) = 0 \quad \forall t \geq \frac{1' (I - P')^{-1} q_0}{\varepsilon}$

with uniform convergence to the origin ε starting in a compact set.