

Fluctuation process

$$\begin{aligned}\hat{V}^r(t) &= \sqrt{r}(\bar{X}^r(t) - \bar{X}(t)) \\ &= \hat{V}^r(0) + \sum_k \hat{N}_k^r(\hat{\Phi}_k^r(t))(\nu'_k - \nu_k) + \sum_k \sqrt{r} \left(\int_0^t (f_k^r(\bar{X}^r(s)) - f_k(\bar{X}^r(s))) ds \right) (\nu'_k - \nu_k) \\ &\quad + \underbrace{\sum_k \left(\int_0^t \nabla f_k(\rho^r(s)) \cdot \hat{V}^r(s) ds \right)}_{(**)} (\nu'_k - \nu_k)\end{aligned}$$

where

$$\hat{\Phi}_k^r(t) = \int_0^t f_k^r(\bar{X}^r(s)) ds$$

Here, $\rho^r(s)$ is between $\bar{X}^r(s)$ and $\bar{X}(s)$.

We know

$$(\hat{V}^r(0), \sum_k \hat{N}_k^r(\hat{\Phi}_k^r(\cdot))(\nu'_k - \nu_k), \bar{X}^r(\cdot)) \Rightarrow (V(0), \sum_k W_k(\Phi_k(\cdot))(\nu'_k - \nu_k), \bar{X}(\cdot))$$

as $r \rightarrow \infty$.

Since $\{\bar{X}^r\}$ converges to $\bar{X}$, which is continuous, we have $\{\bar{X}^r\}$ is C -tight. In particular, for each $T > 0$, and $\epsilon > 0$, there exist K_1, r_1 , such that

$$(*) \quad P(\|\bar{X}^r(\cdot)\|_T > K_1) < \epsilon \quad \forall r \geq r_1.$$

Recall that in the expression of $\hat{V}^r$, f_k^r and f_k are given by the following:

$$\begin{aligned}f_k^r(c) &= \prod_{i=1}^n c_i \left(c_i - \frac{1}{r} \right) \cdots \left(c_i - \frac{\nu_{ik}}{r} + \frac{1}{r} \right) \\ f_k(c) &= \prod_{i=1}^n c_i^{\nu_{ik}}.\end{aligned}$$

It follows that there exist K_2, r_2 such that

$$P\left(\|f_k^r(\bar{X}^r(\cdot)) - f_k(\bar{X}(\cdot))\|_T > \frac{1}{r} K_2\right) < \epsilon, \quad \forall r \geq r_2.$$

Hence,

$$\hat{\Psi}^r = \sum_k \sqrt{r} \int_0^t (f_k^r(\bar{X}^r(s)) - f_k(\bar{X}^r(s))) ds (\nu'_k - \nu_k) \Rightarrow 0 \quad \text{as } r \rightarrow \infty$$

or equivalently, for each $T > 0$, $\epsilon > 0$, there is r_3 such that

$$P(\|\hat{\Psi}^r(\cdot)\|_T > \epsilon) < \epsilon, \quad \forall r \geq r_3.$$

It follows from (*) that there is K_4 and r_4 such that

$$P(\|\nabla f_k(\cdot)\|_T > K_4) < \epsilon, \quad \forall r \geq r_4, \quad \forall k.$$

Our aim now is to show that $\{\hat{V}^r\}$ is C -tight.

Recall Gronwall's inequality: suppose that f, g are real valued functions that are Lebesgue integrable on $[0, T]$ and such that for some $C > 0$,

$$f(t) \leq g(t) + C \int_0^t f(s) ds, \forall t \in [0, T].$$

Then,

$$f(t) \leq g(t) + C \int_0^t e^{C(t-s)} g(s) ds, \forall t \in [0, T].$$

Given $T > 0$ and $\epsilon > 0$ there exist K_5, r_5 such that

$$P(\|\hat{V}^r(0) + \sum_k \hat{N}_k^r(\hat{\Phi}_k^r(\cdot))(\nu'_k - \nu_k) + \hat{\Psi}^r(\cdot)\|_T \leq K_5,$$

$$\|\nabla f_k(\rho^r(\cdot))\|_T \leq \frac{K_5}{(|\nu'_k - \nu_k| + 1)K}, k = 1, \dots, K) \geq 1 - \epsilon$$

when $r \geq r_5$. Call the set of which the probability is taken on the left side above the “good set”. On the good set, we have

$$\begin{aligned} |\hat{V}^r(t)| &\leq K_5 + \left(\sum_{k=1}^K \int_0^t \frac{K_5}{(|\nu'_k - \nu_k| + 1)K} |\hat{V}^r(s)| ds |\nu'_k - \nu_k| \right) \\ &\leq K_5 + \int_0^t K_5 |\hat{V}^r(s)| ds \quad \forall t \in [0, T]. \end{aligned}$$

By Gronwall's inequality, on the same set

$$\begin{aligned} |\hat{V}^r(t)| &\leq K_5 + K_5 \int_0^t e^{K_5(t-s)} K_5 ds \\ &\leq K_6(T) \quad \forall t \in [0, T]. \end{aligned}$$

Hence

$$P(\|\hat{V}^r(\cdot)\|_T \leq K_6(T)) \geq 1 - \epsilon \quad \forall r \geq r_5.$$

Thus, we have the necessary oscillation control of the last term (**) for C -tightness. It follows that $\{\hat{V}^r\}$ is C -tight.

Consider a weak limit point V of $\{\hat{V}^r\}$. Without loss of generality, can assume the convergence of a subsequence of $\{\hat{V}^r\}$ to V is also joint with the same subsequence of $\{(\hat{V}^r(0), \sum_k \hat{N}_k^r(\hat{\Phi}_k^r(\cdot))(\nu'_k - \nu_k), \hat{\Psi}^r(\cdot))\}$. The limit V will satisfy

$$V(t) = V(0) + \sum_k W_k \left(\int_0^t f_k(\bar{X}(s)) ds \right) (\nu'_k - \nu_k) + \sum_k \left(\int_0^t \nabla f_k(\bar{X}(s)) \cdot V(s) ds \right) (\nu'_k - \nu_k),$$

$\forall t \geq 0$.

V is equivalent in law to a solution of

$$\tilde{V}(t) = V(0) + \sum_k \left(\int_0^t \sqrt{f_k(\bar{X}(s))} d\tilde{W}_k(s) \right) (\nu'_k - \nu_k) + \sum_k \left(\int_0^t \nabla f_k(\bar{X}(s)) \cdot \tilde{V}(s) ds \right) (\nu'_k - \nu_k)$$

The S.D.E for $\tilde{V}$ has a unique solution in law. Hence, $\{\tilde{V}^r\}$ in fact converges to V (or $\tilde{V}$).

We have shown that $\sqrt{r}(\bar{X}^r(\cdot) - \bar{X}(\cdot)) \Rightarrow V$ as $r \rightarrow \infty$. We can interpret this as

$$\bar{X}^r(\cdot) - \bar{X}(\cdot) \approx \frac{1}{\sqrt{r}} V.$$

This is called the functional central limit theorem approximation or the Van Kampen approximation. A special case is when $\bar{X}(t) \equiv x_0, \forall t \geq 0$ where x_0 is an equilibrium point for fluid model.

Another Diffusion Approximation

We have

$$\bar{X}^r(t) = \bar{X}^r(0) + \frac{1}{r} \sum_k \hat{N}_k \left(r \int_0^t f_k^r(\bar{X}^r(s)) ds \right) (\nu'_k - \nu_k) + \sum_k \left(\int_0^t f_k^r(\bar{X}^r(s)) ds \right) (\nu'_k - \nu_k)$$

We use $\{\tilde{X}^r\}$ to approximate $\{\bar{X}^r\}$, where

$$\tilde{X}^r(t) = \bar{X}^r(0) + \frac{1}{r} \sum_k W_k \left(r \int_0^t f_k(\tilde{X}^r(s)) ds \right) (\nu'_k - \nu_k) + \sum_k \left(\int_0^t f_k(\tilde{X}^r(s)) ds \right) (\nu'_k - \nu_k)$$

Now $\tilde{W}_k(\cdot) = \frac{W_k(r\cdot)}{\sqrt{r}}$ is a Brownian motion and we can rewrite $\tilde{X}^r$ as:

$$\tilde{X}^r(t) = \tilde{X}^r(0) + \frac{1}{\sqrt{r}} \sum_k \tilde{W}_k \left(\int_0^t f_k(\tilde{X}^r(s)) ds \right) (\nu'_k - \nu_k) + \sum_k \left(\int_0^t f_k(\tilde{X}^r(s)) ds \right) (\nu'_k - \nu_k)$$

From Kurtz's paper, if

$$T_m = \inf\{t \geq 0 : |\bar{X}^r(t)| \vee |\tilde{X}^r(t)| \geq m\}$$

then we have:

$$\sup_{t \in [0, T \wedge T_m]} |\bar{X}^r(t) - \tilde{X}^r(t)| \leq \Gamma_T^r \frac{\log r}{r} \quad a.s$$

The above approximation is often called the Langevin approximation or the diffusion approximation.

Queueing Models

Consider a single server $GI/GI/1$ queue (here GI indicates a sequence of i.i.d. random variables).

$$\xrightarrow{GI} \boxed{\begin{array}{c} FIFO \\ \text{queue} \end{array}} \xrightarrow{GI} \boxed{1} \longrightarrow$$

We let

$$\begin{aligned} Q(t) &= \text{queue length at time } t \\ &= Q(0) + \underbrace{A(t)}_{\text{\# of arrivals up to time } t} - \underbrace{D(t)}_{\text{\# of departures up to time } t} \end{aligned}$$

Here, we rewrite $D(t)$ as $S(T(t))$ where S is the renewal process that goes with the i.i.d service times, and $S(t)$ is the number of service completions that would occur by time t if the server never idles. We let $T(t)$ be the busy time of the server up to time t , and it can be expressed as: $T(t) = \int_0^t 1_{\{Q(s) > 0\}} ds$

From the fluid model, we have:

$$\bar{Q}^r(t) = \frac{Q^r(rt)}{r} = \frac{Q^r(0)}{r} + \frac{A(rt)}{r} - \frac{S(T(rt))}{r}$$

and $\bar{Q}^r \Rightarrow \bar{Q}$. We also write

$$\frac{S(T(rt))}{r} = \frac{S(\frac{rT(rt)}{r})}{r}, \quad \text{and} \quad \bar{T}^r(t) = \frac{T(rt)}{r}$$

$\{\bar{T}^r\}$ is tight. Let $\bar{T}$ be a weak limit point of this sequence. Then we can write

$$\bar{Q}(t) = \bar{Q}(0) + \lambda t - \mu \bar{T}(t) = \bar{Q}(0) + (\lambda - \mu)(t) + \mu \bar{I}(t)$$

where, $I(t)$ is the idle time and $I(t) = t - T(t)$, so that $I^r(t) = \frac{rt}{r} - \frac{T(rt)}{r}$ and $\bar{I}(t) = t - \bar{T}(t)$. Notice $\bar{I}(\cdot)$ increases only when $\bar{Q}(\cdot)$ is zero.