

Diffusion approximation for single class queuing networks

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1 Multi-dimensional Skorohod Problem

$\mathbb{D}_+^d = \{x : [0, \infty) \rightarrow \mathbb{R}^d, x(0) \geq 0\}$.

A pair of paths $(z, y) \in \mathbb{D}_+^d \times \mathbb{D}_+^d = \mathbb{D}_+^{2d}$ is a solution of the Skorohod Problem (SP) for $x \in \mathbb{D}_+^d$ with reflection matrix R if,

(i) $z(t) = x(t) + Ry(t), \quad t \geq 0$

(ii) $z(t) \geq 0, \quad t \geq 0$

(iii) For $i = 1, \dots, d$,

(a) $y_i(0) = 0$

(b) y_i is non-decreasing

(c) y_i can increase only when z_i is zero, i.e.

$$\int_{(0, \infty)} z_i(s) dy_i(s) = 0 \Leftrightarrow \int_{(0, \infty)} \mathbf{1}_{(0, \infty)}(z_i(s)) dy_i(s) = 0$$

1.1 Harrison-Reiman (1981)

Let $R = (I - P')\text{diag}(\mu)$, where P is a transient transition matrix for a MC on d -states, $\mu > 0$.

For each $x \in \mathbb{D}_+^d$, there is a unique pair (z, y) solving the Skorohod Problem. The mapping $x \rightarrow (z, y)$ is continuous where $x \in \mathbb{D}_+^d, (z, y) \in \mathbb{D}_+^{2d}$.

2 Single class queuing network

$$Q_i^r(t) = Q_i^r(0) + E_i(t) - S_i(T_i^r(t)) + \sum_j \Phi_i^j(S_j(T_j^r(t))) \quad (1)$$

Apply diffusion scaling with $\lambda = \mu$:

$$\hat{Q}_i^r(t) = \frac{Q_i^r(rt)}{\sqrt{r}} \quad (2)$$

$$\hat{E}_i^r(t) = \frac{E_i(rt) - \alpha_i rt}{\sqrt{r}} \quad (3)$$

$$\hat{S}_i^r(t) = \frac{S_i(rt) - \mu_i rt}{\sqrt{r}} \quad (4)$$

$$\hat{\Phi}_i^{j,r}(t) = \frac{\Phi_i^j([rt]) - P_{ji}rt}{\sqrt{r}} \quad (5)$$

$$\bar{T}_i^r(t) = \frac{T_i^r(rt)}{r} \quad (6)$$

$$(7)$$

$(\hat{E}^r, \hat{S}^r, \hat{\Phi}^{1,r}, \dots, \hat{\Phi}^{n,r})$ are independent processes and as $r \rightarrow \infty$,

$$(\hat{E}^r, \hat{S}^r, \hat{\Phi}^{1,r}, \dots, \hat{\Phi}^{n,r}) \Rightarrow (\tilde{E}, \tilde{S}, \tilde{\Phi}^1, \dots, \tilde{\Phi}^n).$$

Here $\tilde{E}$ is a n -dimensional Brownian Motion (BM) with covariance matrix $\text{diag}(\alpha_i^3 a_i^2)$ where a_i^2 is the variance of the inter-arrival times for the i^{th} queue, $\tilde{S}$ is an n -dimensional BM with covariance matrix $\text{diag}(\mu_i^3 b_i^2)$, where b_i^2 is the variance of the service times for server i , and $\tilde{\Phi}^j$ is an n -dimensional BM with covariance matrix Υ^j obtained as follows:

$$\Phi^j(m) = \sum_{l=1}^m \phi^j(l) \quad (8)$$

$$E(\Phi_i^j(m)) = mP_{ji}, \quad (9)$$

$$\text{cov}(\phi^j(l))_{kk'} = \begin{cases} \text{var}(\phi_k^j(1)) & k = k'; \\ \underbrace{E(\phi_k^j(1)\phi_{k'}^j(1)) - E(\phi_k^j)E(\phi_{k'}^j)}_0 & k \neq k'. \end{cases} \quad (10)$$

$$= \begin{cases} P_{jk} - P_{jk}^2 & k = k'; \\ -P_{jk}P_{jk'} & k \neq k'. \end{cases} \quad (11)$$

$$= \Upsilon_{kk'}^j. \quad (12)$$

Now,

$$\begin{aligned} \hat{Q}_i^r(t) &= \hat{Q}_i^r(0) + \hat{E}_i^r(t) - \hat{S}_i^r(\bar{T}_i^r(t)) + \sum_j \hat{\Phi}_i^{j,r}(\bar{S}_j^r(\bar{T}_j^r(t))) \\ &\quad + \frac{1}{\sqrt{r}}(\alpha_i rt - \mu_i T_i^r(rt) + \sum_j rP_{ji}\bar{S}_j^r(\bar{T}_j^r(t))) \end{aligned} \quad (13)$$

$$\sqrt{r}\bar{S}_j^r(\bar{T}_j^r(t)) = \hat{S}_j^r(\bar{T}_j^r(t)) + \mu_j \bar{T}_j^r(t)\sqrt{r}. \quad (14)$$

In summary,

$$\begin{aligned}\hat{Q}_i^r(t) &= \hat{Q}_i^r(0) + \hat{E}_i^r(t) - \hat{S}_i^r(\bar{T}_i^r(t)) + \sum_j \hat{\Phi}_i^{j,r}(\bar{S}_j^r(\bar{T}_j^r(t))) + \sum_j P_{ji} \hat{S}_j^r(\bar{T}_j^r(t)) \\ &\quad + \sqrt{r}(\alpha_i t - \mu_i \bar{T}_i^r(t) + \sum_j P_{ji} \mu_j \bar{T}_j^r(t))\end{aligned}\quad (15)$$

$$\bar{I}_i(t) = t - \bar{T}_i^r(t) \quad (16)$$

$$\hat{I}_i^r(t) = \sqrt{r} \bar{I}_i^r(t) \quad (17)$$

Hence,

$$\begin{aligned}\hat{Q}_i^r(t) &= \hat{Q}_i^r(0) + \hat{E}_i^r(t) - \hat{S}_i^r(\bar{T}_i^r(t)) + \sum_j \hat{\Phi}_i^{j,r}(\bar{S}_j^r(\bar{T}_j^r(t))) \\ &\quad + \sum_j P_{ji} \hat{S}_j^r(\bar{T}_j^r(t)) \\ &\quad + \sqrt{r}(\alpha_i t - \mu_i t + \underbrace{\sum_j P_{ji} \mu_j t}_{=0, as \lambda = \alpha + P' \lambda, \lambda = \mu}) \\ &\quad + \sqrt{r}(\mu_i \bar{I}_i^r(t) - \sum_j P_{ji} \mu_j \bar{I}_j^r(t)).\end{aligned}\quad (18)$$

Assume $\hat{Q}^r(0) \Rightarrow 0$. Then,

$$\begin{aligned}\hat{Q}^r(t) &= \hat{Q}^r(0) + \hat{E}^r(t) - (I - P') \hat{S}^r(\bar{T}^r(t)) \\ &\quad + \sum_j \hat{\Phi}^{j,r}(\bar{S}_j^r(\bar{T}_j^r(t))) + (I - P') \text{diag}(\mu) \hat{I}^r(t),\end{aligned}\quad (19)$$

where $\hat{I}_i^r(\cdot)$ increases only when $\hat{Q}_i^r(\cdot)$ is zero. Let

$$\hat{X}^r(t) = \hat{Q}^r(0) + \hat{E}^r(t) - (I - P') \hat{S}^r(\bar{T}^r(t)) + \sum_j \hat{\Phi}^{j,r}(\bar{S}_j^r(\bar{T}_j^r(t))) + (I - P') \text{diag}(\mu) \hat{I}^r(t). \quad (20)$$

Then $(\hat{Q}^r, \hat{I}^r)$ is the solution of the SP for $\hat{X}^r$.

Claim: $\bar{T}^r \Rightarrow 1(\cdot)$, where vector valued function. $\mathbf{1}_i(t) = t \forall i, \forall t$.

Proof. $(\bar{Q}^r, \bar{E}^r, \bar{S}^r, \bar{\Phi}^{1,r}, \dots, \bar{\Phi}^{n,r}, \bar{I}^r)$ is C-tight and any weak limit point is a fluid model solution:

$$\bar{Q}(t) = 0 + (I - P') \text{diag}(\mu) \bar{I}(t).$$

Now, $(\bar{Q}, \bar{I})$ is the unique solution and consequently, $\bar{Q}(t) = 0, \bar{I}(t) = 0$ for all $t \geq 0$, and since $\bar{I}(t) = \mathbf{1}t - \bar{T}(t)$, we have that $\bar{T}(t) = t$ for all $t \geq 0$. $\square$

Combining this with the FCLT convergence of stochastic primitives we may conclude that as $r \rightarrow \infty$,

$$\hat{X}^r \Rightarrow \tilde{X}$$

where, $\tilde{X}(t) = \tilde{E}(t) - (I - P')\tilde{S}(t) + \sum_j \tilde{\Phi}^j(\mu_j t)$

$(\hat{Q}^r, \hat{I}^r) = \psi(\hat{X}^r)$, where ψ is continuous (this follows from SP, where ψ is the mapping from $\hat{X}^r$ to $(\hat{Q}^r, \hat{I}^r)$), where $(\hat{Q}^r, \hat{I}^r)$ is the solution of the SP for $\hat{X}^r$. It follows by the Continuous Mapping Theorem that as $r \rightarrow \infty$,

$$(\hat{Q}^r, \hat{I}^r) \Rightarrow \psi(\tilde{X}) = (\tilde{Q}, \tilde{I}).$$

$\tilde{Q}$ is called the reflecting Brownian motion, while $\tilde{X}$ is the Brownian motion.