

STOCHASTIC MODELS OF COMPLEX NETWORKS

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Lecture 1

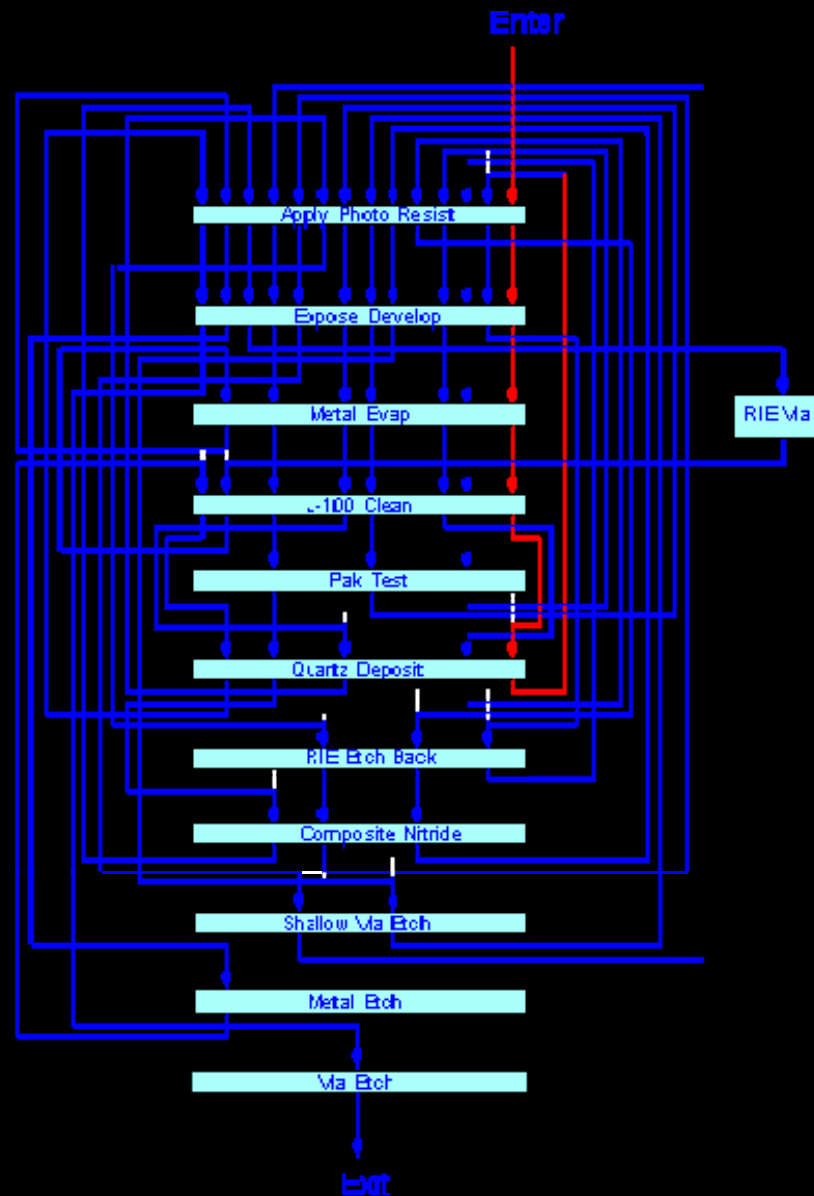
INTRODUCTION

September 29, 2008

APPLICATIONS

- Two main examples of stochastic models of complex networks will be used in this course:
 - Queueing networks which arise in modeling service networks, manufacturing systems, telecommunications and computer systems
 - Biochemical reaction networks which arise in systems biology

Semiconductor Wafer Fab: P. R. Kumar



Call Center: First Direct (branchless retail banking)

Larreche et al., INSEAD '97 (see also Gans, Koole, Mandelbaum '93)



Biochemical Reaction Network

(Mincheva and Craciun, 2008)

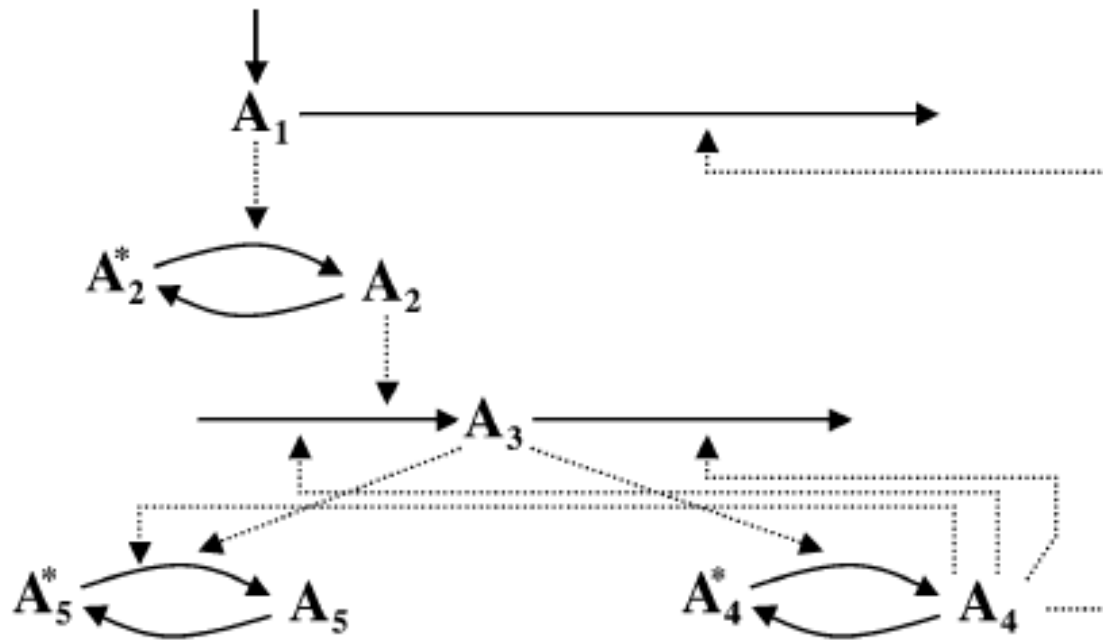


Fig. 1. Cell cycle and apoptosis network studied in [37], [38]. Thick arrows denote mass transfer, and dotted arrows denote an influence of a species on a reaction, without loss of mass for that species.

Biochemical Reaction Network

2

F. J. Brandenburg, M. Forster, A. Pick, M. Raitner, and F. Schreiber

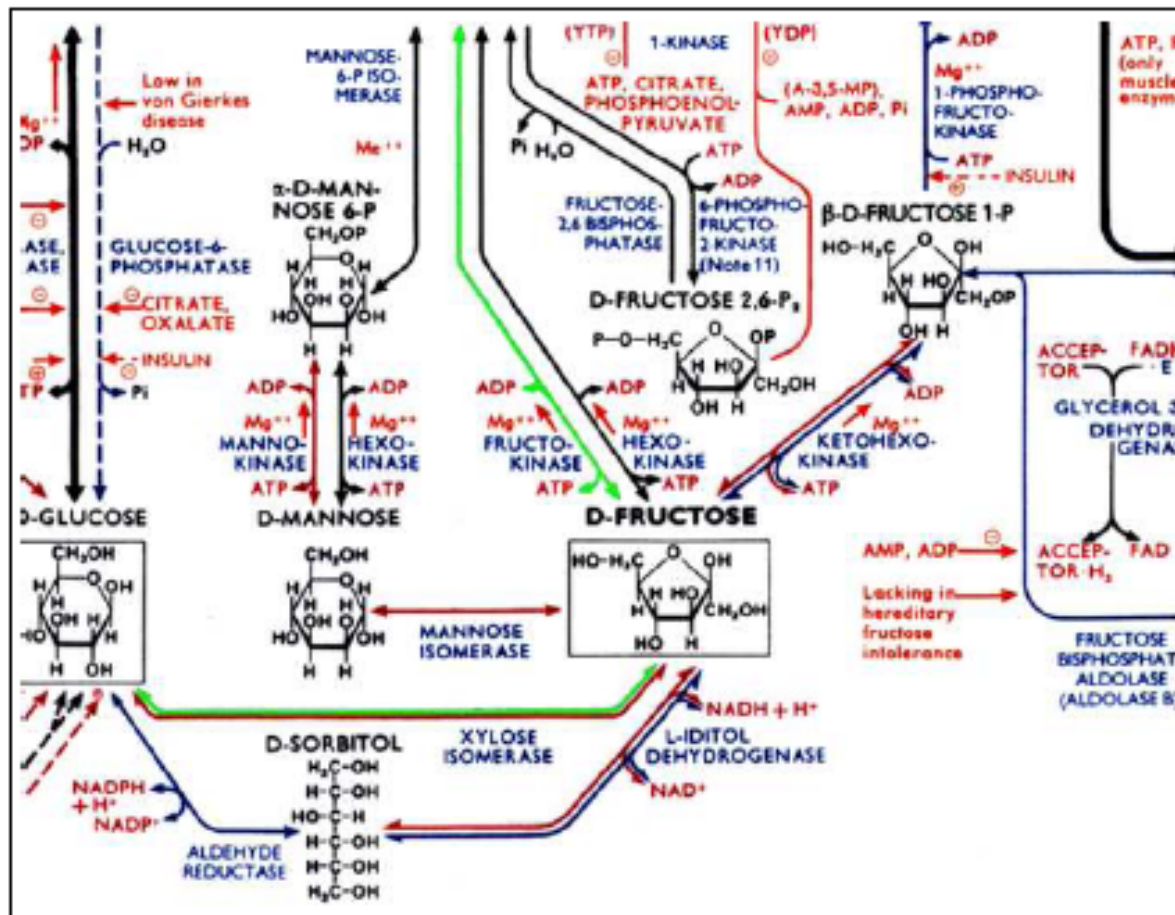


Fig. 1. A cutting of the *Boehringer Biochemical Pathways* poster [16] showing the complexity of the network

Biochemical Reaction Network

BioPath – Exploration and Visualization of Biochemical Pathways

3

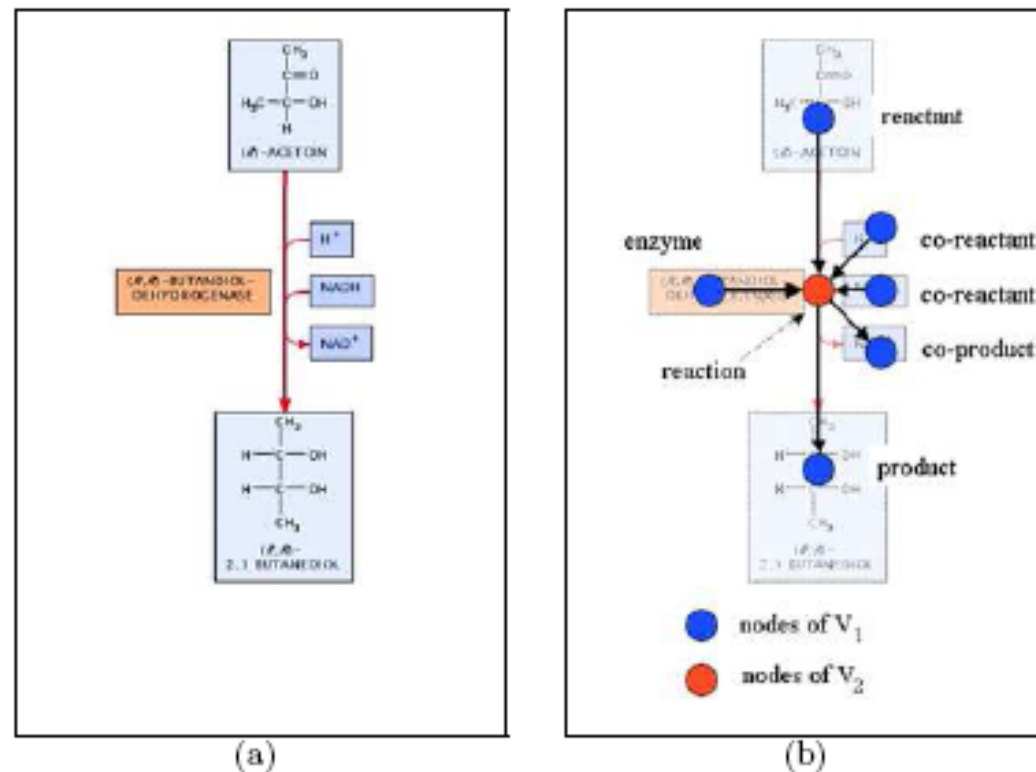


Fig. 2. The diagram (a) shows a biochemical reaction. The connections between enzymes and other compounds are usually displayed by placing an enzyme close to the corresponding hyper-edge. Image (b) shows the representation of this reaction as directed bipartite graph

A Hierarchy of Models

- Microscopic models – usually stochastic, detailed dynamic models; sometimes these are Markov processes but not always; often hard to analyze
- First order deterministic ordinary differential equation models – fluid models
- Second order stochastic differential equation models – often diffusion models

Important subtheme: interplay between levels of modeling

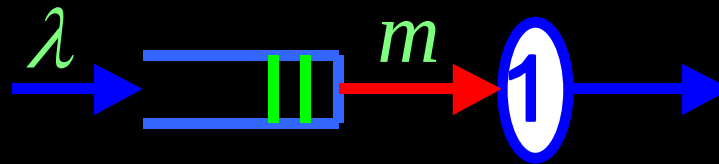
Mathematical Tools

- Fundamental stochastic processes
- Path spaces for stochastic processes
- Weak convergence of stochastic processes
- Invariance principles
- Stationary distributions for Markov processes
- Reversibility and quasireversibility

A SIMPLE EXAMPLE:

SINGLE STATION QUEUE

M/M/1 Queue



- Poisson arrivals at rate λ (independent of service times)
- i.i.d. exponential service times mean m
- FIFO order of service, infinite buffer
- Traffic intensity $\rho = \lambda m$
- Queue length is a birth-death process (Markov)
- Positive recurrent (stable) iff $\rho < 1$
- Stationary distribution $\pi_i = \rho^i (1 - \rho)$, $i = 0, 1, 2, \dots$
- Mean steady-state queue length $L = \rho / (1 - \rho) = \lambda W$

GENERALIZATIONS

- More general interarrival and service times
GI/GI/1
- More servers, infinite server queue as a simple biochemical reaction
- Time dependent arrivals

Lecture 2

Intro (cont.) and path space

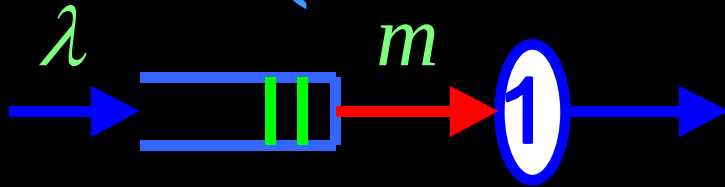
October 1, 2008

APPROXIMATE DYNAMIC MODELS

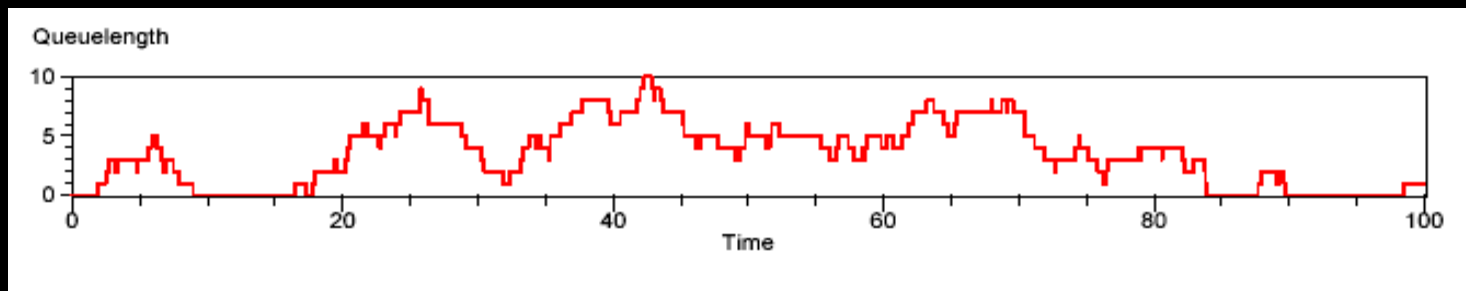
- Most stochastic network models cannot be analyzed exactly
- Consider approximate models (valid under some scaling limit, e.g., heavily loaded, many sources, many servers, large networks)
- Two main classes of approximate models:
 - Fluid models (functional law of large numbers)
 - Diffusion models (functional central limit theorem)

M/M/1 Queue

(Simulation of Dynamics)

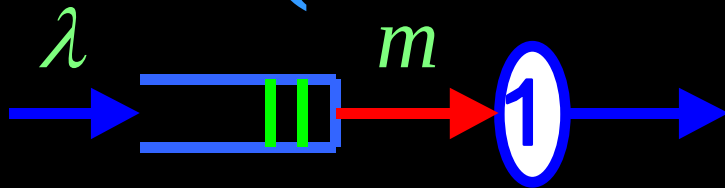


$$\rho = \lambda = 0.9524$$

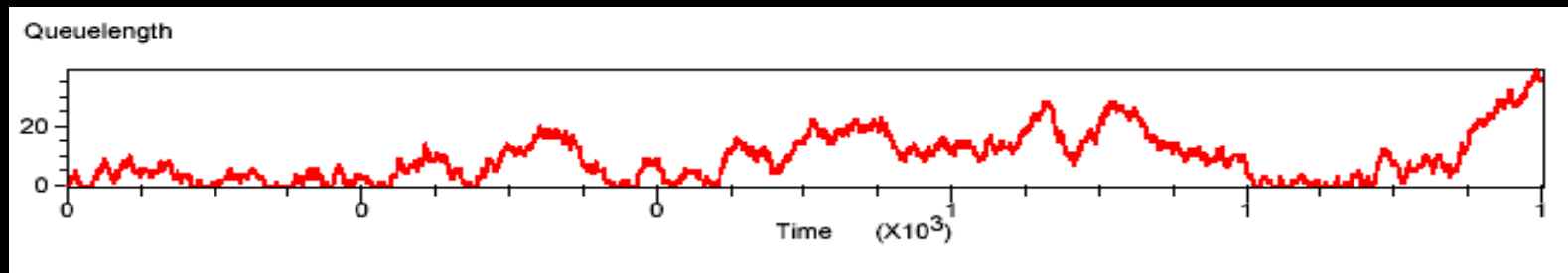
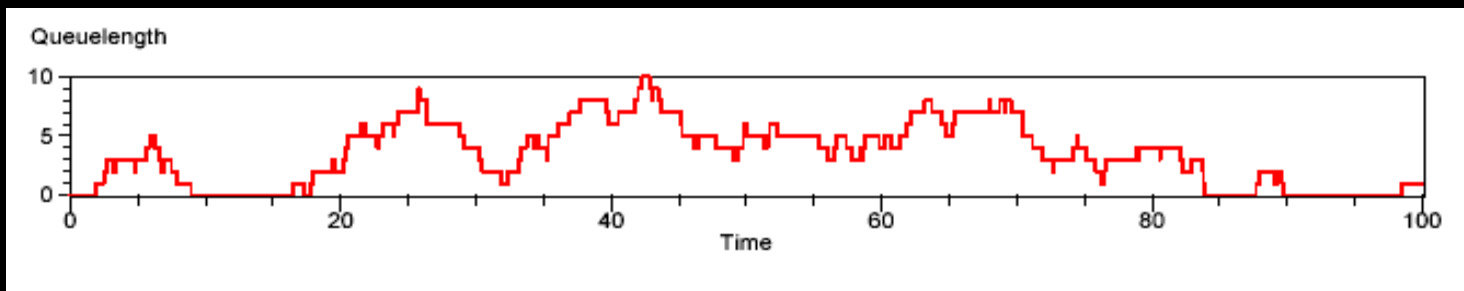


M/M/1 Queue

(Simulation of Dynamics)

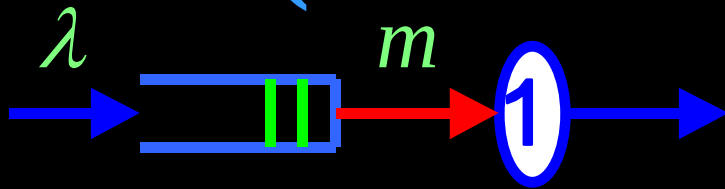


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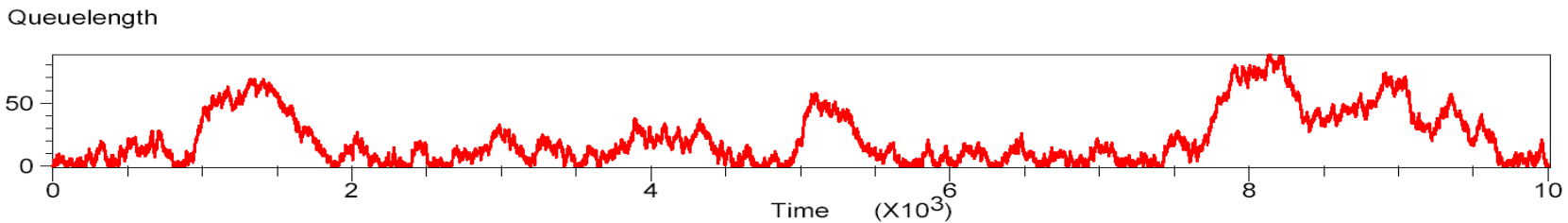
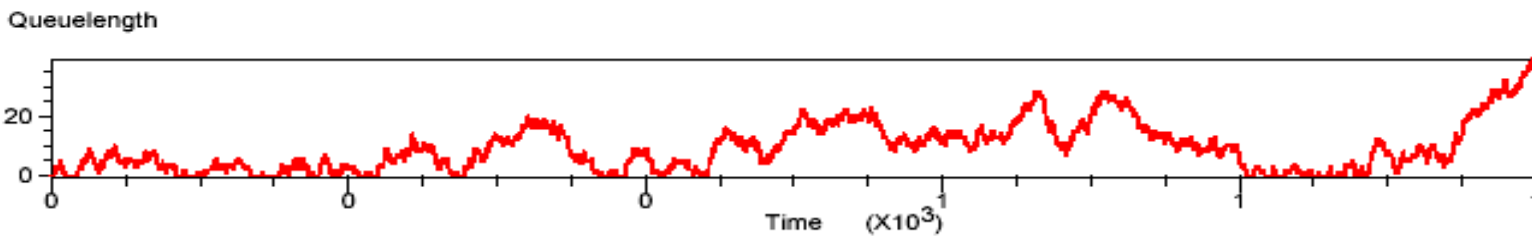
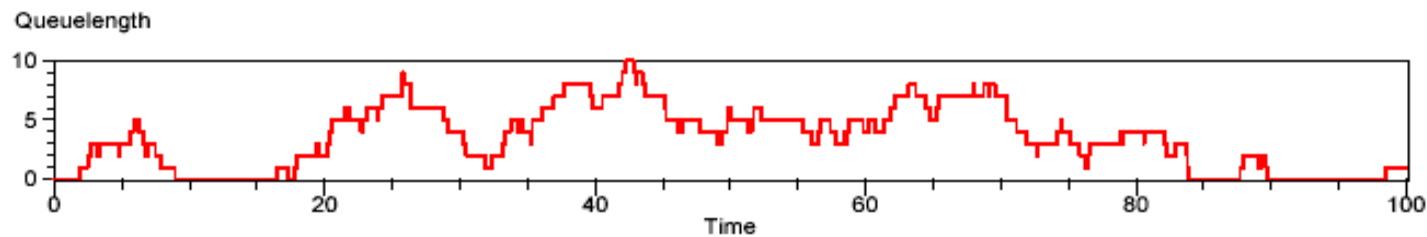


M/M/1 Queue

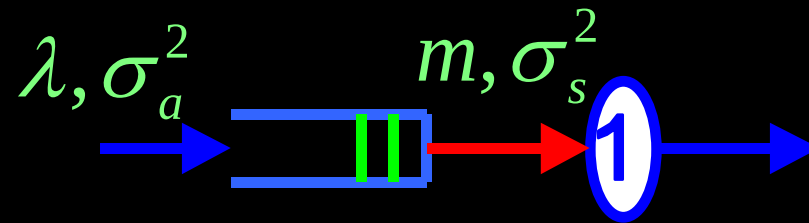
(Simulation of Dynamics)



$$\rho = \lambda = 0.9524$$



GI/GI/1 Queue (Dynamics)



$Q(t)$ = queue length at time t

Start system empty (for simplicity)

Theorem (A. Borovkov '67, Iglehart-Whitt '70): For $\rho \approx 1$,

$$(1 - \rho)Q(\cdot / (1 - \rho)^2) \approx Q^*(\cdot) \quad \text{where } Q^*(\cdot)$$

is a one-dimensional reflecting Brownian motion

with drift $-m^{-1}$ and variance parameter $\lambda^3 \sigma_a^2 + m^{-3} \sigma_s^2$

One-dimensional Reflecting Brownian Motion

$$Q^*(t) = X^*(t) + Y^*(t)$$

$$Y^*(t) = \sup\{-X^*(s) : 0 \leq s \leq t\}$$

X^* = Brownian motion

