

Heavy Tail FCLT Notes

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This is a continuation from Alex Bilik's lecture.

Normal Domain of Attraction of a Stable Law [3]

We start with the following definition: a cdf F is in the *normal domain of attraction* of a stable law if there are constants c and m_n such that $(cn^{1/\alpha})^{-1}(S_n - m_n) \Rightarrow S_\alpha(1, \beta, 0)$, where $S_n = \sum_{i=1}^n X_i$ and $\{X_i\}$ are i.i.d. with distribution function F . Then we have the following heavy-tail central limit theorem which provides a relatively clean approximation of S_n 's distribution:

Theorem 7. *Let $\{X_n : n \geq 1\}$ be an iid sequence with cdf F . The cdf belongs to the normal domain of attraction of $S_\alpha(1, \beta, 0)$ for $0 < \alpha < 2$ iff both*

$$G^c(x) \sim Ax^{-\alpha} \text{ as } x \rightarrow \infty \quad (1)$$

for positive constants A and α , and

$$\frac{F^c(x)}{G^c(x)} \rightarrow \frac{1+\beta}{2} \text{ as } x \rightarrow \infty.$$

Then c can be chosen to be $(A/C_\alpha)^{1/\alpha}$, where

$$C_\alpha := \begin{cases} \frac{1-\alpha}{\Gamma(2-\alpha)\cos(\pi\alpha/2)} & \text{if } \alpha \neq 1 \\ 2/\pi & \text{if } \alpha = 1 \end{cases}$$

and (A, α) are from (1). (This C_α definition also applies to Theorem 6.)

This is easy to prove given Theorem 6. c_n there can be chosen to be of the form $cn^{1/\alpha}$ while satisfying $\lim_{n \rightarrow \infty} \frac{nL(c_n)}{c_n^\alpha} = C_\alpha$ iff $L(x)$ approaches a constant as $x \rightarrow \infty$. So, a cdf belongs to the normal domain of attraction of a stable law of index α iff $x^\alpha G^c(x) = L(x)$ holds for $L(x) \rightarrow$ some constant A , and $\frac{F^c(x)}{G^c(x)} \rightarrow \frac{1+\beta}{2}$. Dividing both sides of the first equation by x^α yields the equivalent condition $G^c(x) \sim Ax^{-\alpha}$, and then setting $c_n = (A/C_\alpha)^{1/\alpha}$ ensures $\lim_{n \rightarrow \infty} \frac{nL(c_n)}{c_n^\alpha} = C_\alpha$.

Stable Lévy Motion [3]

Definition: A *Lévy process* is a stochastic process $\mathbb{L} \equiv \{\mathbb{L}(t) : t \geq 0\}$ that has sample paths in $\mathbb{D}$, $\mathbb{L}(0) = 0$, and stationary and independent increments.

Definition: A *standard stable Lévy motion* is a Lévy process $\mathbb{S} \equiv \{\mathbb{S}(t) : t \geq 0\}$ where the increments have stable laws. Specifically,

$$\mathbb{S}(t+s) - \mathbb{S}(s) \stackrel{d}{=} S_\alpha(t^{1/\alpha}, \beta, 0) \stackrel{d}{=} t^{1/\alpha} S_\alpha(1, \beta, 0) \quad \forall s \geq 0, t > 0$$

for some $\alpha \in (0, 2]$ and $\beta \in [-1, 1]$. This is also called a “standard (α, β) -stable Lévy motion.” (Note that when $\alpha = 2$, this is just Brownian motion.)

Except when $\alpha = 1$ and $\beta \neq 1$, stable Lévy motion is self-similar with index $1/\alpha$, i.e.

$$\{\mathbb{S}(ct) : t \geq 0\} \stackrel{d}{=} \{c^{1/\alpha} \mathbb{S}(t) : t \geq 0\}.$$

A few qualitative properties of non-Brownian ($\alpha < 2$) stable Lévy motion:

- Except for some deterministic drift, it is a “pure jump process,” with infinitely many discontinuities in any positive length finite interval a.s. (Fortunately, it can still be made to have sample paths in $\mathbb{D}$.)
- For $\beta = 1$ (i.e. skewed entirely to the right), it is nondecreasing for $\alpha < 1$. (This is called a “stable subordinator.”) With $\alpha \geq 1$, it’s a combination of negative drift and only positive jumps.
- When $\alpha > 1$, there are sample paths with unbounded variation in each bounded interval (like Brownian motion).

Stable FCLT

Skorokhod demonstrated in [2] that all ordinary CLTs have FCLT counterparts in $(\mathbb{D}, J_1)$. A proof for the theorem to follow, under the assumption the X^n ’s are semimartingales, is in [1] (Theorem VII.3.4).

Theorem 8. [3] *Under the conditions of Theorem 6, plus $c_n^{-1}(S_n - m_n) \xrightarrow{d} S_\alpha(1, \beta, 0)$ in $\mathbb{R}$, we have*

$$\mathbf{S}_n \xrightarrow{d} \mathbf{S} \text{ in } (\mathbb{D}, J_1)$$

for the associated normalized process

$$\mathbf{S}_n(t) := c_n^{-1}(S_{\lfloor nt \rfloor} - m_n t), t \geq 0,$$

where the limit $\mathbf{S}$ is a standard (α, β) -stable Lévy motion.

We conclude with some illustrations of heavy-tail process behavior. The first three slides are centered random walks with Pareto steps $U_k^{-2/3}$, where each U_k is drawn uniformly from $[0, 1]$, and they converge to stable Lévy motion with $\alpha = 3/2$. The last three slides are uncentered random walks with Pareto steps U_k^{-2} , and they converge to stable Lévy motion with $\alpha = 1/2$ – note that it is common here for a single step to account for more than half the total variation, regardless of the length of the random walk.

References

- [1] J. Jacod, A. N. Shiryaev, "Limit Theorems for Stochastic Processes." Springer-Verlag, 1987.
- [2] A. V. Skorokhod, "Limit theorems for stochastic processes with independent increments," *Theor. Probability Appl.* 2, 138-171.
- [3] W. Whitt, "Stochastic-Process Limits." Springer, 2002.