

Heavy-Tail Central Limit Theorem

Alex Bilik

November 14, 2008

First, here are three alternate, equivalent definitions of a *stable distribution*:

Definition 1 (taken from [2]). A random variable X is said to have a stable distribution if for any positive numbers A and B , there is a positive number C and a real number D such that

$$AX_1 + BX_2 \stackrel{d}{=} CX + D$$

where X_1 and X_2 are independent copies of X .

Definition 2 (also from [2]). A random variable X is said to have a stable distribution if for any $n \geq 2$, there is a positive number C_n and a real number D_n such that

$$X_1 + X_2 + \cdots + X_n \stackrel{d}{=} C_n X + D_n$$

where $X_1, X_2, \dots, X_n$ are independent copies of X .

Definition 3 (also from [2]). A random variable X is said to have a stable distribution if there are parameters $0 < \alpha \leq 2, \sigma \geq 0, -1 \leq \beta \leq 1$, and μ real such that its characteristic function $f(\theta)$ has the following form:

$$\log f(\theta) = \begin{cases} -\sigma^\alpha |\theta|^\alpha (1 - i\beta(\operatorname{sgn} \theta) \tan \frac{\pi\alpha}{2}) + i\mu\theta & \text{if } \alpha \neq 1 \\ -\sigma |\theta| (1 + i\beta \frac{2}{\pi} (\operatorname{sgn} \theta) \ln |\theta|) + i\mu\theta & \text{if } \alpha = 1. \end{cases}$$

The parameter α is the index of stability, β is a skewness parameter, σ a scale parameter, and μ is a location parameter.

We denote such a distribution $S_\alpha(\sigma, \beta, \mu)$. We have the following properties:

Proposition 1 ([2]). Let X_1, X_2 be independent random variables with $X_i \sim S_\alpha(\sigma_i, \beta_i, \mu_i)$, $i = 1, 2$. Then $X_1 + X_2 \sim S_\alpha(\sigma, \beta, \mu)$, with

$$\sigma = (\sigma_1^\alpha + \sigma_2^\alpha)^{1/\alpha}, \beta = \frac{\beta_1 \sigma_1^\alpha + \beta_2 \sigma_2^\alpha}{\sigma_1^\alpha + \sigma_2^\alpha}, \mu = \mu_1 + \mu_2.$$

Proposition 2 ([2]). Let $X \sim S_\alpha(\sigma, \beta, \mu)$ and let a be a real constant. Then $X + a \sim S_\alpha(\sigma, \beta, \mu + a)$.

Proposition 3 ([2]). Let $X \sim S_\alpha(\sigma, \beta, \mu)$ and let a be a non-zero constant. Then

$$\begin{aligned} aX &\sim S_\alpha(|a|\sigma, \text{sgn}(a)\beta, a\mu) \quad \text{if } \alpha \neq 1 \\ aX &\sim S_\alpha(|a|\sigma, \text{sgn}(a)\beta, a\mu - \frac{2}{\pi}a(\ln |a|)\sigma\beta) \quad \text{if } \alpha = 1 \end{aligned}$$

Proposition 4 ([2]). For any $0 < \alpha < 2$,

$$X \sim S_\alpha(\sigma, \beta, 0) \iff -X \sim S_\alpha(\sigma, -\beta, 0).$$

Since μ and σ are centering and scaling parameters, we can characterize the most important features of a stable distribution by the two parameters α and β ; that is, we can focus on distributions of the form $S_\alpha(1, \beta, 0)$.

We are interested in stable distributions because they are limits of sums of iid random variables, as expressed by the following:

Definition 4 ([5]). A distribution function F is said to be in the *domain of attraction* of a stable law with exponent $\alpha < 2$ if there are constants $A_n > 0, B_n$ such that when $X_1, X_2, \dots$, are i.i.d. random variables with distribution F and $S_n = \sum_{i=1}^n X_i$, then

$$\frac{S_n}{A_n} - B_n \xrightarrow{d} X$$

and X is a stable random variable with exponent α .

This leads to one version of the heavy-tail central limit theorem:

Theorem 5 ([5]). A distribution function F is in the domain of attraction of a stable law with exponent $\alpha < 2$ if and only if there are constants $M^+, M^- \geq 0, M^+ + M^- > 0$, such that as $y \rightarrow \infty$:

$$\lim \frac{F(-y)}{1 - F(y)} = \frac{M^-}{M^+}$$

and for every $\xi > 0$,

$$\begin{aligned} M^+ > 0 &\Rightarrow \lim_{y \rightarrow \infty} \frac{1 - F(\xi y)}{1 - F(y)} = \frac{1}{\xi^\alpha} \\ M^- > 0 &\Rightarrow \lim_{y \rightarrow -\infty} \frac{F(-\xi y)}{F(-y)} = \frac{1}{\xi^\alpha} \end{aligned}$$

The first condition compares how the two tails of a one-dimensional distribution decay with respect to each other, the second looks at the decay rate of each tail by itself.

Here is another version of the theorem with a more concrete presentation. First, we introduce a new definition:

Definition 5 ([4]). A function L defined on $(0, \infty)$ is called *regularly varying* with index α if

$$\lim_{x \rightarrow \infty} \frac{L(\xi x)}{L(x)} = \xi^\alpha.$$

A function L defined on $(0, \infty)$ is called *slowly varying* if

$$\lim_{x \rightarrow \infty} \frac{L(\xi x)}{L(x)} = 1.$$

Suppose X_1 is a r.v. with distribution function F . Let F^c denote the complementary cdf for F and let G^c denote the *complementary cdf* for $|X_1|$ given by

$$G^c \equiv P(|X_1| > x) = 1 - F(x) + F(-x).$$

In the following version of the theorem, we use the notation

$$c_n^{-1}(S_n - m_n) \xrightarrow{d} S_\alpha(\sigma, \beta, \mu)$$

to denote convergence to a stable law. We then get the following version of the heavy tail central limit theorem:

Theorem 6 ([4] stable-law CLT). Let $\{X_n : n \geq 1\}$ be an i.i.d. sequence of real-valued random variables with cdf F . The cdf F belongs to the domain of attraction of $S_\alpha(1, \beta, 0)$ for $0 < \alpha < 2$ if and only if

$$x^\alpha G^c(x) = L(x)$$

where L is slowly varying, and

$$\frac{F^c(x)}{G^c(x)} \rightarrow \frac{1+\beta}{2} \quad \text{as } x \rightarrow \infty$$

The space-scaling constants c_n then must satisfy

$$\lim_{n \rightarrow \infty} \frac{nL(c_n)}{c_n^\alpha} = C_\alpha,$$

for

$$C_\alpha = \left(\int_0^\infty x^{-\alpha} \sin x \, dx \right)^{-1}$$

and the translation constants m_n may be chosen to satisfy

$$m_n = \begin{cases} 0 & \text{if } 0 < \alpha < 1 \\ nc_n \int_{-\infty}^\infty \sin(x/c_n) \, dF(x) & \text{if } \alpha = 1 \\ n \int_{-\infty}^\infty x \, dF(x) & \text{if } 1 < \alpha < 2. \end{cases}$$

In this case, $c_n = n^{1/\alpha} L_0(n)$, where L_0 is slowly varying (in general different from L).

References

- [1] William Feller *An Introduction to Probability Theory and its Applications* Wiley, 1971
- [2] Gennady Samorodnitsky and Murad Taqqu *Stable Non-Gaussian Random Processes* Chapman & Hall, 1994
- [3] B. V. Gnedenko & A. N. Kolmogorov *Limit Distributions for Sums of Independent Random Variables* Addison-Wesley, 1968
- [4] Ward Whitt *Stochastic-Process Limits* Springer, 2002
- [5] Leo Breiman *Probability* Addison-Wesley, 1968