

October 20, 2008

289A Class Notes.

Recall: A sequence of stochastic processes $(X_n)_{n=1}^{\infty}$ (assume sample paths are in $\mathbb{D}^d$) converges in distribution to a d -dimensional stochastic process $X \Leftrightarrow$ the associated laws $(\pi^n)_{n=1}^{\infty}$ converge weakly to the law, π , of X ; i.e.

$$\forall f \in \mathcal{C}_b(\mathbb{D}^d), \quad \int_{\mathbb{D}^d} f d\pi^n \xrightarrow{n \rightarrow \infty} \int_{\mathbb{D}^d} f d\pi \quad (*)$$

Note: Continuity is with respect to metric induced by J_1 -Skorohod topology.

A "friendlier" version of $(*)$ is:

$$E[f(X^n)] \xrightarrow{n \rightarrow \infty} E[f(X)] \quad \forall f \in \mathcal{C}_b(\mathbb{D}^d).$$

Skorohod Representation Theorem

Suppose $(\pi^n)_{n=1}^{\infty}, \pi$ are probability measures on $(M, \mathcal{M})$ (where M is a Polish space and $\mathcal{M}$ is Borel σ -algebra of M). Such that

$$\pi^n \xrightarrow{w} \pi \quad (\text{in above sense})$$

Then there exist a probability space $(\Omega, \mathcal{F}, P)$ and M -valued random entities (stochastic processes, if $M = \mathbb{D}^d$) $(X^n)_{n=1}^{\infty}$ and X ($X^n, X: \Omega \rightarrow M$) such that

i) $\forall n, X^n$ has law π^n .

X has law π .

ii) $X^n \rightarrow X$ P -a.s. as $n \rightarrow \infty$; i.e., for P -a.e. $\omega \in \Omega$, $X^n(\omega) \rightarrow X(\omega)$ as $n \rightarrow \infty$. ■

Note: $X^n(\omega), X(\omega) \in M$, which has a metric, so that convergence makes sense.

Remark: We could start with a sequence of stochastic processes $(X^n)_{n=1}^{\infty}$ defined on different probability spaces, and use Skorohod to redefine $(\tilde{X}^n)$ on some space Ω , so that we get a.s. convergence instead of just convergence in distribution.

Consequence:

Continuous Mapping Theorem (in the context of $\mathbb{D}^d$).

For fixed, positive integers d and k , let $(h_n)_{n=1}^{\infty}$ and h be measurable functions from $(\mathbb{D}^d, \mathcal{D}^d)$ into $(\mathbb{D}^k, \mathcal{D}^k)$. (Simplest case: $h_n = h \ \forall n$). Define the continuity set of h

$$C_h = \left\{ x \in \mathbb{D}^d \mid \begin{array}{l} h_n(x_n) \rightarrow h(x) \text{ in } \mathbb{D}^k \text{ whenever} \\ x_n \rightarrow x \text{ in } \mathbb{D}^d \end{array} \right\}$$

Let $(X^n)_{n=1}^{\infty}, X$ be d -dimensional stochastic processes such that $\bullet X^n \Rightarrow X$ as $n \rightarrow \infty$

$$\bullet P(X \in C_h) = 1.$$

Then $h_n(X^n) \rightarrow h(X)$ as $n \rightarrow \infty$. ■

Example

$$d = k = 1$$

$$h_n = h \ \forall n$$

$$h(x)(t) = \max_{0 \leq s \leq t} |x(s)| \quad \forall t \geq 0.$$

Claim: $h: \mathbb{D}^1 \rightarrow \mathbb{D}^1$ is continuous

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proof on next
page

PROOF: Assume $x_n \rightarrow x$ in $\mathbb{D}$; i.e. $\forall T > 0 \exists \lambda \exists (\gamma_n) \subset \Gamma$ s.t.

P.3

$$① \sup_{t \in [0, T]} |\gamma_n(t) - t| \xrightarrow{n \rightarrow \infty} 0$$

$$② \sup_{t \in [0, T]} |x_n(\gamma_n(t)) - x(t)| \xrightarrow{n \rightarrow \infty} 0$$

To show $h_n(x_n) \rightarrow h(x)$ in $\mathbb{D}$, we let $(\gamma_n)_{n=1}^{\infty}$ be as above. We must show that

$$\sup_{t \in [0, T]} |h_n(x_n(\gamma_n(t))) - h(x(t))| \xrightarrow{n \rightarrow \infty} 0 \quad \forall T > 0.$$

Well,

$$\begin{aligned} & \sup_{t \in [0, T]} |h_n(x_n(\gamma_n(t))) - h(x(t))| \\ &= \sup_{t \in [0, T]} \left| \max_{0 \leq s \leq t} |x_n(\gamma_n(s))| - \max_{0 \leq s \leq t} |x(s)| \right| \\ &= \| \|x_n \circ \gamma_n\|_t - \|x\|_t \|_T \\ &\leq \|x_n \circ \gamma_n - x\|_T \xrightarrow{n \rightarrow \infty} 0 \quad \text{by } ②. \quad \blacksquare \end{aligned}$$

PROOF (of the CMT)

p.4

By the Skorohod Rep. Thm, $\exists$ processes $(\tilde{X}^n)_{n=1}^\infty$ and $\tilde{X}$ st.

$$X^n \stackrel{d}{=} \tilde{X}^n \quad \forall n$$

$$X \stackrel{d}{=} \tilde{X}$$

$$\tilde{X}^n \rightarrow \tilde{X} \text{ a.s. as } n \rightarrow \infty.$$

$\therefore$ for a.e. $\omega \in \Omega$, $\tilde{X}^n(\omega) \rightarrow \tilde{X}(\omega)$ in $\mathbb{D}^d$ as $n \rightarrow \infty$.

Note that since $\tilde{X} \stackrel{d}{=} X$, $P(\tilde{X} \in C_h) = P(X \in C_h) = 1$. ; i.e.

for a.e. ω , $\tilde{X}(\omega) \in C_h$

By defn. of C_h , we have that for a.e. ω ,

$$h_n(\tilde{X}^n(\omega)) \rightarrow h(\tilde{X}(\omega)) \text{ as } n \rightarrow \infty.$$

$$\text{i.e. } h_n \circ \tilde{X}^n \rightarrow h \circ \tilde{X} \text{ a.s.} \quad (*)$$

Let $f \in C_b(\mathbb{D}^k)$.

Since f is continuous, $(*) \Rightarrow f(h_n(\tilde{X}^n)) \xrightarrow{n \rightarrow \infty} f(h(\tilde{X}))$ a.s.

By the Bounded Convergence Thm (since f is bdd),

$$E[f(h_n(\tilde{X}^n))] \rightarrow E[f(h(\tilde{X}))] \text{ as } n \rightarrow \infty$$

$$\therefore E[f(h_n(X^n))] \rightarrow E[f(h(X))] \text{ as } n \rightarrow \infty$$

$$\text{since } \tilde{X}^n \stackrel{d}{=} X^n, \tilde{X} \stackrel{d}{=} X.$$

$\therefore h_n(X^n) \Rightarrow h(X)$ as $n \rightarrow \infty$ since f was arbitrary. (in $C_b(\mathbb{D}^k)$) ■

Random Time Change Theorem

P. 5

Suppose that $(X^n)_{n=1}^\infty$ and X are d -dimensional stochastic processes.

Let $D_0 = \{x: [0, \infty) \rightarrow [0, \infty) \mid x \text{ is nondecreasing}\}$

Let $(\underline{\Phi}_n)_{n=1}^\infty, \underline{\Phi}$ be stochastic processes whose sample paths are in D_0 (i.e., they are random time changes).

Suppose • $(X^n, \underline{\Phi}^n) \Rightarrow (X, \underline{\Phi})$ as $n \rightarrow \infty$

thought of as a path in $\mathbb{D}^{d+1}$

• $P(X \text{ has continuous paths}) = 1$

Then $X^n \circ \underline{\Phi}^n \Rightarrow X \circ \underline{\Phi}$ as $n \rightarrow \infty$ ▀

Proof (using the C.M.T.):

Define $\psi = x \circ \phi$ for $x \in \mathbb{D}^d, \phi \in D_0 \rightarrow \psi(x, \phi)(t) = x(\phi(t))$.

(Think of ψ as "h" from the C.M.T.) So we have

$$\psi: \underbrace{\mathbb{D}^d \times D_0}_{\subset \mathbb{D}^{d+1}} \rightarrow \mathbb{D}^d$$

Claim: ψ is continuous whenever X is.

Suppose $(x_n, \phi_n) \rightarrow (x, \phi)$ in $\mathbb{D}^d \times D_0$, where the topology in $\mathbb{D}^d \times D_0$ is not the product topology, but rather the topology on $\mathbb{D}^{d+1}$, and $x \in C^d$ w.p. 1.

Must show: $\psi(x_n, \phi_n) \xrightarrow{n \rightarrow \infty} \psi(x, \phi)$ in $\mathbb{D}^d$.

We have that $x_n \rightarrow x$ unif. on compacta as $n \rightarrow \infty$ (since x is continuous with probability 1)

By our definition of convergence in $\mathbb{D}^d$, we have for each $T > 0, \exists (\delta_n)_{n \geq 1} : \|\varphi_n(\gamma_n(\cdot)) - \varphi(\cdot)\|_T \rightarrow 0$ as $n \rightarrow \infty$. ^{P. 6} $\forall \epsilon > 0, \exists \delta > 0$ s.t. $\|\gamma_n(\cdot) - \gamma(\cdot)\|_T \rightarrow 0$ as $n \rightarrow \infty$.

Therefore $\exists n_1$ s.t.

$$\|\varphi_n(\gamma_n(\cdot))\|_T \leq \underbrace{\|\varphi(\cdot)\|_T + 1}_{:= K} \quad \forall n \geq n_1$$

FOR $n \geq n_1$,

$$\begin{aligned} & \|\gamma_n(\varphi_n(\gamma_n(\cdot))) - \gamma(\varphi(\cdot))\|_T \\ & \leq \|\gamma_n(\varphi_n(\gamma_n(\cdot))) - \gamma(\varphi_n(\gamma_n(\cdot)))\|_T \\ & \quad + \|\gamma(\varphi_n(\gamma_n(\cdot))) - \gamma(\varphi(\cdot))\|_T \quad (\Delta\text{-inequality}) \end{aligned}$$

$$\leq \sup_{0 \leq s \leq K} |\gamma_n(s) - \gamma(s)| + \|\gamma(\varphi_n(\gamma_n(\cdot))) - \gamma(\varphi(\cdot))\|_T$$

$\downarrow n \rightarrow \infty$
 0

$\downarrow n \rightarrow \infty$
 0

since $\gamma_n \rightarrow \gamma$ uniformly on compacta

since γ is continuous and therefore uniformly continuous over compact time intervals.

◦◦ by the Continuous Mapping Thm,

$$\Psi(\gamma^n, \mathbb{I}^n) \rightarrow \Psi(\gamma, \mathbb{I}) \text{ as } n \rightarrow \infty.$$



Back to Tightness (of Probability Measures on $\mathbb{D}^d$):

P. 7

Question: How do we prove sequences converge in distribution in the first place?

Recall: $(\pi^n)_{n=1}^\infty$ is tight on the Polish space $(M, \mathcal{M})$ if
 $\forall \varepsilon > 0 \exists A \subset M, \text{cpct}, \text{ such that}$
$$\pi^n(A) \geq 1 - \varepsilon \quad \forall n.$$

e.g. $M = \mathbb{D}^d$

(had CRITERION for $A \subset \mathbb{D}^d$ to be compact).

THEOREM: A sequence of probability measures $(\pi^n)_{n=1}^\infty$ on $(\mathbb{D}^d, \mathcal{D}^d)$ is tight iff $\forall T > 0$ and $\forall \varepsilon > 0$

$$\text{i) } \lim_{K \rightarrow \infty} \limsup_{n \rightarrow \infty} \pi^n(\{x \in \mathbb{D}^d \mid \|x\|_T \geq K\}) = 0$$

$$\text{ii) } \lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} \pi^n(\{x \in \mathbb{D}^d \mid \omega'(x, \delta, T) \geq \varepsilon\}) = 0. \quad \blacksquare$$