

MATH 286: Lecture 1

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1 Introduction

In this lecture, we will have a brief overview of different ways in which stochastic differential equations can arise and be viewed. Notational note: for a process X , X_t and $X(t)$ are used interchangeably.

1.1 Ways to think about SDEs

1.1.1 Randomly Perturbed Dynamical Systems

Roughly, we can think of a stochastic differential equation (SDE) as an ordinary differential equation (ODE) with an added random perturbation in the dynamics. For example, if $X = \{X_t, t \geq 0\}$ is a real valued process describing the state of a system at each time t , a stochastic differential equation governs the time evolution of this process if X satisfies the following equation in some sense:

$$\frac{dX_t}{dt} = b(t, X_t) + \sigma(t, X_t)W_t, \quad t \in (0, \infty), \quad (1)$$

where $W = \{W_t, t \geq 0\}$ is so-called “white noise”. When σ depends on the state, the additional noise in the dynamics is called *multiplicative noise*.

In the equation (1), $\frac{dX_t}{dt}$ and W_t are yet to be defined rigorously. Now, we would “like” $\{W_t, t \geq 0\}$ to be a continuous time analog of a certain “sequence” of *i.i.d* mean zero random variables having the following properties:

- (i) W_t is independent of W_s whenever $t \neq s$,
- (ii) $E\{W(t)\} = 0$ for each $t \geq 0$,
- (iii) stationarity: $(W_{t_1+t}, \dots, W_{t_n+t})$ has the same distribution as $(W_{t_1}, \dots, W_{t_n})$ for any $t, t_1 < t_2 < \dots < t_n$ in $[0, \infty)$,

- (iv) W_t is a generalized Gaussian random variable with variance $+\infty$, for each $t \geq 0$.

Unfortunately, there is no real valued stochastic process with these properties. If there were such a process, by extrapolating from the spectral theory for Gaussian processes, one might expect that, for each $t \geq 0$, the covariance function $\theta(t) = E\{W_0 W_t\} = \int_{\mathbb{R}} e^{i\lambda t} f(\lambda) d\lambda$, where f is the so-called “spectral density”. The second and fourth properties above would imply that, for each $t > 0$, $\theta(t) = 0$, and $\theta(0) = +\infty$, which in turn would suggest that the spectral density f is constant. This constant density function is why W is called *white noise* (all of the frequencies are equally represented).

In order to circumvent the aforementioned difficulty, we will resort to an integral version of the equation (1):

$$X_t = X_0 + \int_0^t b(s, X_s) ds + \int_0^t \sigma(s, X_s) dB_s, \quad (2)$$

where $B = \{B_t, t \geq 0\}$ is a Brownian motion. Formally, we are thinking that

$$dB_t = W_t dt.$$

We know that this is only true in an intuitive sense because a Brownian motion is almost surely nowhere differentiable, i.e., for almost every realization of a Brownian motion path, $t \rightarrow B(t)$ is not differentiable for all t .

Example 1: *Noisy Bonhoeffer-van der Pol oscillator.* A noisy 2-dimensional simplification of the 4-dimensional Hodgkin-Huxley system for modeling the firing of a single neuron. Questions of interest include qualitative behavior and the effect of increasing noise. See scanned transparencies.

Aside on colored noise: Some scientists want to use colored noise rather than white noise in their dynamical models of stochastic systems. One common way to achieve this is by replacing the white noise W by a process V satisfying the SDE:

$$V_t = V_0 - \int_0^t c V_s ds + \sigma B_t, \quad (3)$$

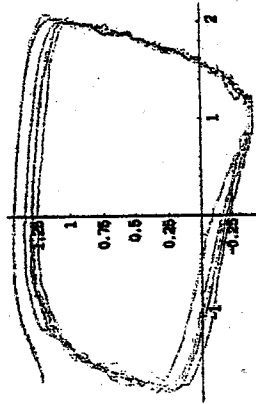
for positive constants c, σ . One might write this equation informally as

$$\frac{dV_t}{dt} = -cV_t + \sigma W_t. \quad (4)$$

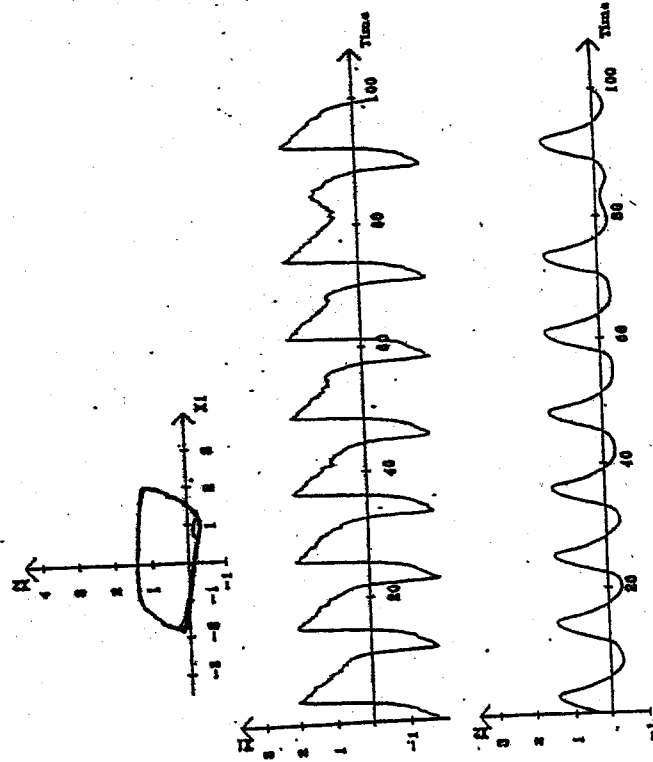
This process V is called an Ornstein-Uhlenbeck process and is well defined as a continuous stochastic process and models involving it can in principle be manipulated using the theory that we will develop.

NOISY BONHOEFFER-VAN DER POL OSCILLATOR

- **BONHOEFFER-VAN DER POL MODEL**
 - 2-dimensional simplification of 4-dimensional Hodgkin-Huxley system of ordinary differential equations for modeling firing of a single neuron
 - effects of membrane imperfections and firing of nearby neurons can be simulated by the addition of low intensity noise in the voltage dynamics



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Bonhoeffer-Van der Pol oscillator, weak noise $\sigma = 0.1$

NOISY BONHOEFFER-VAN DER POL OSCILLATOR

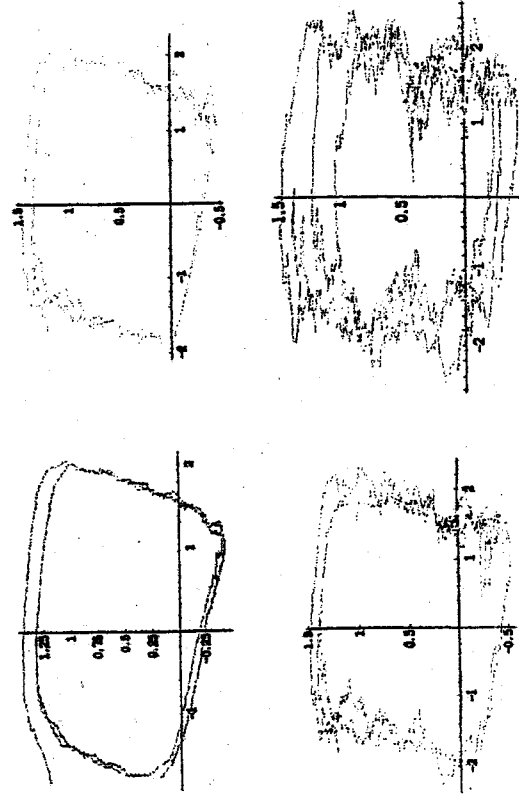
X^1 = membrane voltage X^2 = membrane permeability

$$dX_t^1 = c(X_t^1 + X_t^2 - \frac{1}{3}(X_t^1)^3 + z)dt + \sigma dB_t$$

$$dX_t^2 = -\frac{1}{c}(X_t^1 + bX_t^2 - a)dt$$

- **Parameters:** $a = 0.7$, $b = 0.8$, $c = 3.0$, $z = -0.34$
- **Limit cycle with $\sigma = 0$**

Effect of increase in noise



Example 2: Models from mathematical finance. If X_t represents the value of a simple bond at time t , Y_t represents the value of a stock at time t , and $0 < r < \mu$, $\sigma > 0$, the Black-Scholes model proposes that X, Y satisfy

$$dX_t = rX_t dt$$

and

$$dY_t = \mu Y_t dt + \sigma Y_t dB_t,$$

for $t \geq 0$. A solution of the second equation is called a geometric Brownian motion. More advanced variants of this model, called *stochastic volatility* models, allow the volatility parameter σ to be a stochastic process itself. One model that has been considered for this is where σ is replaced by a process $\{\sigma_t, t \geq 0\}$ satisfying

$$d\sigma_t = \kappa(\nu - \sigma_t)dt + \theta d\tilde{B}_t, \quad (5)$$

where κ, ν, θ are positive constants and $(B, \tilde{B})$ is a two-dimensional Brownian motion, perhaps with correlated components. Then $\log \sigma_t$ is a “mean-reverting” variant of the Ornstein-Uhlenbeck process.

1.1.2 PDEs, Kolmogorov and Fokker-Planck equations

Another way to view SDEs, which is employed often by physicists, is to consider the distribution of a solution of an SDE at each time t , i.e., to consider the marginal distribution at time t . For example, under suitable conditions, a solution of (2) is a diffusion process (continuous strong Markov process), and then one may be interested in studying the process through functions of the form:

$$u_t(s, x) = E[f(X_t) | X_s = x], \text{ for } 0 < s < t < \infty, x \in \mathbb{R}.$$

Under certain “nice” conditions on the process X and the function f , the function $u_t(s, x)$ satisfies

$$0 = \frac{\partial u_t(s, x)}{\partial s} + b(s, x) \frac{\partial u_t(s, x)}{\partial x} + \frac{1}{2} \sigma^2(s, x) \frac{\partial^2 u_t(s, x)}{\partial x^2} \quad (6)$$

for each $0 < s < t$ and $x \in \mathbb{R}$. The parabolic partial differential equation (6) is referred to as a backward Kolomogorov equation (by mathematicians). There are related adjoint equations called variously Kolmogorov forward equations or Fokker-Planck equations or the diffusion approximation to a chemical master equation.

For us, an essential tool for analyzing SDEs will be Itô’s formula. In particular, this will allow us to connect solutions of equations like (2) with partial differential equations like (3).

1.1.3 Infinitesimal Characteristics

Yet another way to view SDEs is via their infinitesimal characteristics. (This is common in developing approximations to discrete models, e.g., those arising in population genetics.) Under mild conditions, the infinitesimal drift and variance associated with the solution of an SDE are given by

$$b(t, x) = \lim_{h \rightarrow 0} \frac{1}{h} E [X(t+h) - X(t) | X(t) = x] \quad (7)$$

$$\sigma^2(t, x) = \lim_{h \rightarrow 0} \frac{1}{h} E [(X(t+h) - X(t))^2 | X(t) = x]. \quad (8)$$

The quantity $\sigma(t, x)$ is often referred to as the infinitesimal volatility or dispersion. Informally, the idea of these characteristics is that they describe the infinitesimal dynamics of the solution of the process X :

$$X(t+h) \approx X(t) + b(t, X(t))h + \sigma(t, X(t))(B(t+h) - B(t)). \quad (9)$$

The book, *A Second Course on Stochastic Processes* by Karlin and Taylor, has many examples of diffusion approximations to discrete models arising in applications (see Chapter 15).

2 Aspects of SDEs

- Modeling via physical intuition or approximation of a discrete system.
- Well posedness: stochastic integrals, existence and uniqueness of solutions.
- Analysis: connections to pde, stability, numerical approximation.
- Applications.