

MATH 285, SPRING 2019
HOMEWORK 6, due Tuesday, June 11, 2019, 5pm in TA homework box
THIS HOMEWORK IS OPTIONAL

1. Consider an experiment where a biased coin is flipped 3 times. The probability of a head is $1/4$ and of a tail is $3/4$ on each trial and the trials are assumed to be independent. Let X_i denote the outcome of the i^{th} coin flip where X_i is two if a head is obtained and zero if a tail is obtained. Let $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and let $\mathcal{F}_n = \sigma\{X_i : i = 1, \dots, n\}$, for $n = 1, 2, 3$. Let $S_0 = 0$ and $S_n = \sum_{i=1}^n X_i$, $n = 1, 2, 3$.

- (a) For each $n = 0, 1, 2, 3$, determine the partition of Ω that generates $\mathcal{F}_n$.
- (b) Determine the following conditional expectations (remember these should be random variables).

$$E[X_3 | \mathcal{F}_2]$$

$$E[S_3 | \mathcal{F}_1]$$

- (c) Find a martingale associated with the sequence $\{X_n, n = 1, 2, 3\}$ and with $\{S_n^2, n = 0, 1, 2, 3\}$ (the last part is harder).

2. Consider a random variable X with finite mean and variance and let $\mathcal{H} \subset \mathcal{G}$ be sub- σ -algebras of $\mathcal{F}$, the σ -algebra with respect to which all random variables are measurable. Prove the total conditional variance formula:

$$\text{Var}(X | \mathcal{H}) = E[\text{Var}(X | \mathcal{G}) | \mathcal{H}] + \text{Var}(E[X | \mathcal{G}] | \mathcal{H})$$

where for any σ -algebra $\mathcal{K} \subset \mathcal{F}$,

$$\text{Var}(X | \mathcal{K}) = E[X^2 | \mathcal{K}] - (E[X | \mathcal{K}])^2$$

3. Consider a biased simple random walk on the integers with probability $0 < p < 1/2$ of moving to the right by one and probability $q = 1 - p$ of moving to the left by one, at each step. Let S_n denote the position of the random walk at time n , for $n = 0, 1, 2, \dots$. Then we may represent S_n by

$$S_n = x + \sum_{i=1}^n \xi_i, \quad n = 0, 1, \dots,$$

where x is the initial state, $\{\xi_i\}_{i=1}^\infty$ are i.i.d. random variables with $P(\xi_i = +1) = p$ and $P(\xi_i = -1) = q$ for each i . Suppose that $0 < x < b$ where x and b are positive integers.

- (a) Show that $M_n = S_n + (q - p)n$, $n = 0, 1, 2, \dots$ is a martingale with respect to the filtration generated by S_n , $n = 0, 1, 2, \dots$.
- (b) Show that $M_n = (q/p)^{S_n}$, $n = 0, 1, 2, \dots$, is a martingale with respect to its own filtration.
- (c) Let $T = \inf\{n \geq 0 : S_n = 0 \text{ or } b\}$. Show that T is a stopping time relative to the filtration generated by $\{S_n, n = 0, 1, 2, \dots\}$.
- (d) Use the martingale in part (a) and the optional stopping theorem to show that $E[T] < \infty$.
- (e) Use the optional stopping theorem and your answer to parts (b) and (d) to find the probability that the random walk reaches 0 before b .

4. Suppose that $X = \{X_t, t \geq 0\}$ is standard one-dimensional Brownian motion. Fix $a > 0$. Prove (using the definition) that $\{X(at)/\sqrt{a}, t \geq 0\}$ is also a standard one-dimensional Brownian motion.

5. Let $X = \{X_t, t \geq 0\}$ be a standard one-dimensional Brownian motion. Compute the conditional probability,

$$P(X_2 > 0 | X_1 > 0),$$

and

$$E[X_s X_t]$$

for $0 \leq s \leq t$.