

so $\alpha = 1/5$, $\beta = 4/5$, $\gamma = -2/5$ and

$$p_{11}^{(n)} = \frac{1}{5} + \left(\frac{1}{2}\right)^n \left\{ \frac{4}{5} \cos \frac{n\pi}{2} - \frac{2}{5} \sin \frac{n\pi}{2} \right\}.$$

More generally, the following method may in principle be used to find a formula for $p_{ij}^{(n)}$ for any M -state chain and any states i and j .

- (i) Compute the eigenvalues $\lambda_1, \dots, \lambda_M$ of P by solving the characteristic equation.
- (ii) If the eigenvalues are distinct then $p_{ij}^{(n)}$ has the form

$$p_{ij}^{(n)} = a_1 \lambda_1^n + \dots + a_M \lambda_M^n$$

for some constants $a_1, \dots, a_M$ (depending on i and j). If an eigenvalue λ is repeated (once, say) then the general form includes the term $(an + b)\lambda^n$.

- (iii) As roots of a polynomial with real coefficients, complex eigenvalues will come in conjugate pairs and these are best written using sine and cosine, as in the example.

Exercises

1.1.1 Let $B_1, B_2, \dots$ be disjoint events with $\bigcup_{n=1}^{\infty} B_n = \Omega$. Show that if A is another event and $\mathbb{P}(A|B_n) = p$ for all n then $\mathbb{P}(A) = p$.

Deduce that if X and Y are discrete random variables then the following are equivalent:

- (a) X and Y are independent;
- (b) the conditional distribution of X given $Y = y$ is independent of y .

1.1.2 Suppose that $(X_n)_{n \geq 0}$ is Markov (λ, P) . If $Y_n = X_{kn}$, show that $(Y_n)_{n \geq 0}$ is Markov (λ, P^k) .

1.1.3 Let X_0 be a random variable with values in a countable set I . Let $Y_1, Y_2, \dots$ be a sequence of independent random variables, uniformly distributed on $[0, 1]$. Suppose we are given a function

$$G : I \times [0, 1] \rightarrow I$$

and define inductively

$$X_{n+1} = G(X_n, Y_{n+1}).$$

Show that $(X_n)_{n \geq 0}$ is a Markov chain and express its transition matrix P in terms of G . Can all Markov chains be realized in this way? How would you simulate a Markov chain using a computer?

Suppose now that $Z_0, Z_1, \dots$ are independent, identically distributed random variables such that $Z_i = 1$ with probability p and $Z_i = 0$ with probability $1 - p$. Set $S_0 = 0$, $S_n = Z_1 + \dots + Z_n$. In each of the following cases determine whether $(X_n)_{n \geq 0}$ is a Markov chain:

- (a) $X_n = Z_n$, (b) $X_n = S_n$,
 (c) $X_n = S_0 + \dots + S_n$, (d) $X_n = (S_n, S_0 + \dots + S_n)$.

In the cases where $(X_n)_{n \geq 0}$ is a Markov chain find its state-space and transition matrix, and in the cases where it is not a Markov chain give an example where $P(X_{n+1} = i | X_n = j, X_{n-1} = k)$ is not independent of k .

1.1.4 A flea hops about at random on the vertices of a triangle, with all jumps equally likely. Find the probability that after n hops the flea is back where it started.

A second flea also hops about on the vertices of a triangle, but this flea is twice as likely to jump clockwise as anticlockwise. What is the probability that after n hops this second flea is back where it started? [Recall that $e^{\pm i\pi/6} = \sqrt{3}/2 \pm i/2$.]

1.1.5 A die is 'fixed' so that each time it is rolled the score cannot be the same as the preceding score, all other scores having probability $1/5$. If the first score is 6, what is the probability p that the n th score is 6? What is the probability that the n th score is 1?

Suppose now that a new die is produced which cannot score one greater (mod 6) than the preceding score, all other scores having equal probability. By considering the relationship between the two dice find the value of p for the new die.

1.1.6 An octopus is trained to choose object A from a pair of objects A, B by being given repeated trials in which it is shown both and is rewarded with food if it chooses A . The octopus may be in one of three states of mind: in state 1 it cannot remember which object is rewarded and is equally likely to choose either; in state 2 it remembers and chooses A but may forget again; in state 3 it remembers and chooses A and never forgets. After each trial it may change its state of mind according to the transition matrix

State 1	$\frac{1}{2}$	$\frac{1}{2}$	0
State 2	$\frac{1}{2}$	$\frac{1}{12}$	$\frac{5}{12}$
State 3	0	0	1.

Someone suggests that the record of successive choices (a sequence of A s and B s) might arise from a two-state Markov chain with constant transition probabilities. Discuss, with reference to the value of $P_{n+1}(A)$ that you have found, whether this is possible.

1.1.7 Let $(X_n)_{n \geq 0}$ be a Markov chain on $\{1, 2, 3\}$ with transition matrix

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 2/3 & 1/3 \\ p & 1-p & 0 \end{pmatrix}.$$

Calculate $\mathbb{P}(X_n = 1 | X_0 = 1)$ in each of the following cases: (a) $p = 1/16$, (b) $p = 1/6$, (c) $p = 1/12$.

1.2 Class structure

It is sometimes possible to break a Markov chain into smaller pieces, each of which is relatively easy to understand, and which together give an understanding of the whole. This is done by identifying the communicating classes of the chain.

We say that i leads to j and write $i \rightarrow j$ if

$$\mathbb{P}_i(X_n = j \text{ for some } n \geq 0) > 0.$$

We say i communicates with j and write $i \leftrightarrow j$ if both $i \rightarrow j$ and $j \rightarrow i$.

Theorem 1.2.1. For distinct states i and j the following are equivalent:

- (i) $i \rightarrow j$;
- (ii) $p_{i_0 i_1} p_{i_1 i_2} \cdots p_{i_{n-1} i_n} > 0$ for some states $i_0, i_1, \dots, i_n$ with $i_0 = i$ and $i_n = j$;
- (iii) $p_{ij}^{(n)} > 0$ for some $n \geq 0$.

Proof. Observe that

$$p_{ij}^{(n)} \leq \mathbb{P}_i(X_n = j \text{ for some } n \geq 0) \leq \sum_{n=0}^{\infty} p_{ij}^{(n)}$$

which proves the equivalence of (i) and (iii). Also

$$p_{ij}^{(n)} = \sum_{i_1, \dots, i_{n-1}} p_{ii_1} p_{i_1 i_2} \cdots p_{i_{n-1} j}$$

so that (ii) and (iii) are equivalent. $\square$