

Lecture 8

04/25/2006

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Homework #3 Due May 11 (on web page).

(problem B handed out in class)

↑ handout for

REVERSIBLE MC's

MC is reversible if

$\exists$ probability distribution π on states, such that

$$\pi_i P_{ij} = \pi_j P_{ji} \quad \forall i, j \quad [\text{Detailed balance}]$$

Suppose MC is irreducible and positive recurrent
(so it has a unique stationary distribution)

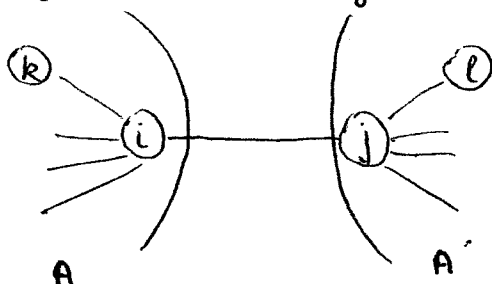
Consider a graph G whose vertices are states of MC
and there is an edge between two states i & j iff
either $P_{ij} > 0$ or $P_{ji} > 0$

Theorem: If G is a tree then MC is reversible

Proof: Let π be the stationary distribution
fix $i \neq j$

if there is no edge between i & j then $\pi_i P_{ij} = \pi_j P_{ji}$ holds.

So suppose there is an edge between i & j .



$$A \cup A' = S$$

Let A = set of all states linked to i
(with $i \leftrightarrow j$ edge severed)

$$A' = S \setminus A$$

$$j \in A'$$

Note $P_{kl} = 0 \quad \forall k \in A \setminus \{i\}, l \in A'$

$$+ P_{lk} = 0 \quad \forall l \in A' \setminus \{j\}, k \in A$$

$$+ P_{il} = 0 \quad \forall l \in A' \setminus \{j\}$$

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Since π is a stationary distribution:

$$\sum_k \pi_k P_{kl} = \pi_l \quad \text{for all } l \in A'$$

$$\Leftrightarrow \pi_i P_{il} + \sum_{k \in S \setminus \{i\}} \pi_k P_{kl} = \pi_l \quad \forall l \in A'$$

$$\Leftrightarrow \pi_i P_{il} + \sum_{k \in A'} \pi_k P_{kl} = \pi_l \underbrace{\sum_{k \in S} P_{lk}}_{=1}$$

Sum over all $l \in A'$

$$\sum_{l \in A'} \pi_i P_{il} + \sum_{l \in A'} \sum_{k \in A'} \pi_k P_{kl}$$

$$= \sum_{l \in A'} \pi_l \sum_{k \in S} P_{lk}$$

$$= \pi_j \sum_{k \in S} P_{jk} + \sum_{l \in A' \setminus \{j\}} \pi_l \sum_{k \in S} P_{lk}$$

$$\pi_i P_{ij} + \sum_{l \in A'} \sum_{k \in A'} \pi_k P_{kl} \quad (\text{since } P_{il} = 0 \text{ for } l \neq j, l \in A')$$

$$= \pi_j P_{ji} + \pi_j \sum_{k \in A'} P_{jk} + \sum_{l \in A' \setminus \{j\}} \pi_l \sum_{k \in A'} P_{lk}$$

$$\cancel{\pi_i P_{ij}} + \cancel{\sum_{l \in A'} \sum_{k \in A'} \pi_k P_{kl}} = \pi_j P_{ji} + \sum_{l \in A'} \sum_{k \in A'} \pi_l P_{lk}$$

$$\Rightarrow \pi_i P_{ij} = \pi_j P_{ji}$$

□

Ref: Book - Reversibility and stochastic Networks. F.P. Kelly.

Metropolis - Hastings Algorithm (cont.).

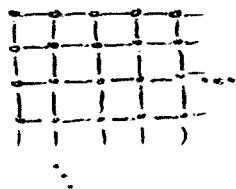
Given π - probability distribution

Q - symmetric transition matrix

$$P_{ij} = Q_{ij} \min(1, \pi_j / \pi_i)$$

Example Ising model in 2-D

$N \times N$ lattice, each site has a spin ± 1



model states as $N \times N$ matrices

with entries taking values in $\{-1, +1\}$

$\Rightarrow 2^{N^2}$ states (configurations)

Energy of a configuration M :

$$H(M) = - \sum_{\substack{(i,j) \\ \sim (i',j')}} M_{ij} M_{i'j'}$$

$(i,j) \sim (i',j')$
↑
neighbors

probability distribution π

inversely proportional to

a is the temperature

$$\pi_a(M) = \frac{\exp(-a H(M))}{\sum_{M' \in S} \exp(-a H(M'))} \quad \left. \vphantom{\sum_{M' \in S}} \right\} \begin{array}{l} \text{partition function} \\ \text{(normalization constant)} \end{array}$$

To use Metropolis - Hastings algorithm, choose:

Q :

$$Q(M, M') = \begin{cases} 1/N^2 \\ 0 \end{cases}$$

if M, M' are neighbors, i.e. M' is obtained from M by changing one entry only in M .

otherwise

⇒ Metropolis - Hastings

$$P_{M,M'} = P(M, M')$$

$$= Q(M, M') \min(1, \pi_a(M')/\pi_a(M))$$

$$\text{Nas, } \frac{\pi_a(M')}{\pi_a(M)} = \frac{\exp(-2a \sum_{(i,j) \sim (i',j')} M'_{ij} M'_{i'j'})}{\exp(-2a \sum_{(i,j) \sim (i',j')} M_{ij} M_{i'j'})}$$

(i,j) is fixed - the site you pick, going from M to M'
to flip in
 (for $Q(M, M') > 0$)

HIDDEN MARKOV MODELS

Markov chains where only observe a function (possibly a random function of the state).

History

- Basic Theory: Baum et al. in 1966-72

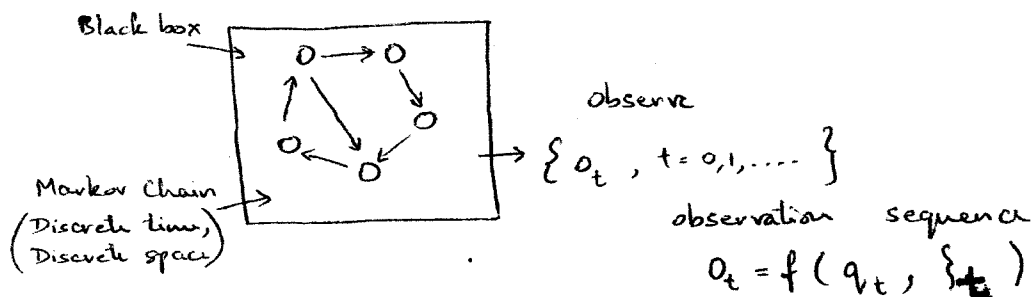
(statistical estimation for probabilistic functions of markov chains)

- Application to speech recognition - Baker (CMU) 1975

- Jerninck et al. (IBM) 1970's

- DNA sequence alignment - Churchill (1989), Borodovsky et al (late 80's)

- Matlab toolbox for HMM



where $\{z_t\}$ = random variable with distribution determined by q_t
 (independent of all other r.v.'s)

N states for m.c. , M outputs possible.

$$P(O_t = Y_k \mid q_t = s_j) = b_{jk} = b_j(k)$$

possible outputs $Y_1, \dots, Y_M$

possible m.c. states $s_1, \dots, s_N$.

Parameters for the Model

$$a = (a_{ij})$$

$$b = (b_{jk})$$

$$\pi = (\pi_1, \dots, \pi_N) = \text{initial distribution for MC.}$$

$$\Rightarrow \lambda = (a, b, \pi)$$

Note: $\{(q_t, O_t), t=0, 1, 2, \dots\}$ is also MC.

$$\begin{aligned} \text{verification } P((q_{t+1}, o_{t+1}) = (s_j, Y_k) \mid (q_t, o_t) = (s_i, Y_i), \dots, (q_0, o_0)) \\ = P((q_{t+1}, o_{t+1}) = (s_j, Y_k) \mid q_t = s_i) \\ = a_{ij} b_{jk} \end{aligned}$$

Example

① Occasionally dishonest casino.

Two dice - one fair and one loaded

$q_t = 1$ or 2 as fair or loaded die is used

$o_t = \text{outcome of the die of type } q_t$

