

APRIL 18, '05

6

Long run behavior of Markov Chains (cont.)

6.1 Quick review of last time

1. Steady-state distribution (SSD)
2. Stationary Distn.
(not every stationary distn is s-s distn)
3. Communicating Classes (CC)
(divide state space into equivalence classes of communicating states)
4. Irreducible Markov Chain: one CC
5. Period of a state i of Markov Chain: $d(i)$ = greatest common divisor of $\{n \geq 1 : P_{ii}^n > 0\}$. If $P_{ii}^{(n)} = 0$ for all $n \geq 1$ then define $d(i) = 0$.

Theorem 6.1.1 Suppose that $P^{(n)} > 0$ for some $n \geq 1$. Then $d(i) = 1$ for all i .

PROOF. Fix a state i and let j be a state such that $P_{ij} > 0$. By assump-

18

6. LONG RUN BEHAVIOR OF MARKOV CHAINS

tion, there is $n \geq 1$ such that $P_{ji}^n > 0$. Then, $P_{ii}^{n+1} = \sum_k P_{ik} P_{ki}^n \geq P_{ij} P_{ji}^n > 0$. Since the greatest common divisor of n and $n+1$ is one, it follows that $d(i) = 1$.

Fact: All states in the same communicating class have the same period.

6.2 Recurrence and Transience

A state i is *recurrent* iff starting from i , the MC returns to i in finite time, with probability 1. In other words, if X is the MC, then i is recurrent iff $P(X_n = i \text{ for some } n \geq 1 | X_0 = i) = 1$. Let $T_i = \inf\{n \geq 1 : X_n = i\}$, the first return time to i . Then i is recurrent iff $P(T_i < \infty | X_0 = i) = 1$.

A state i is *transient* iff it is not recurrent. In other words, i is transient iff $P(X_n = i \text{ for some } n \geq 1 | X_0 = i) < 1$ or equivalently, $P(T_i = \infty | X_0 = i) > 0$.

Using a regeneration argument (based on each return to i), one can show that for a

Recurrent State i : the MC returns to i infinitely many times

Transient State i : the MC returns to i only finitely many times (although the number of returns is a random variable).

If the state space is countably infinite, then recurrent states break into two classes: null recurrent: $E_i[T_i] = \infty$, and positive recurrent: $E_i[T_i] < \infty$.

If the state space is finite, all recurrent states are positive recurrent.

Example Reflected random walk on $\mathbb{N}$.

Let p be the probability of increasing by one step and q be the probability of decreasing by one step (except when at zero, in which case q is the probability of staying at zero at the next step). Here $p + q = 1$ and $0 < p < 1$.

$p > q$: all states are transient

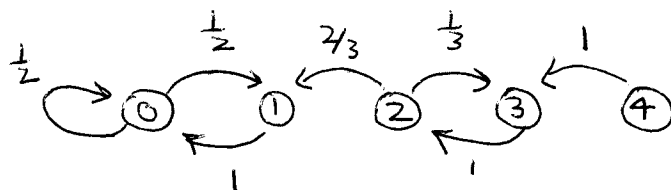
$p = q$: null recurrent

$p < q$: positive recurrent



Fact: All states in a given communicating class are either transient, null recurrent, or positive recurrent.

Example



C.C.	{0, 1}	{2, 3}	{4}
Recc./Trans.	R	T	T

Decompose Transition Matrix:

$$P = \left(\begin{array}{ccc|c} \boxed{P_1} & & & 0 \\ & \boxed{P_2} & & \\ & & \ddots & \\ \hline & S & & Q \end{array} \right)$$

(1st set of rows: rec.; 2nd set of rows: trans.; 1st set of cols: rec.; 2nd set of cols: trans;)

$$P^n = \left(\begin{array}{ccc|c} \boxed{P_1^n} & & & 0 \\ & \boxed{P_2^n} & & \\ & & \ddots & \\ \hline & S_n & & Q^n \end{array} \right)$$

Theorem 6.2.1 BASIC LIMIT THM for MCs For a positive recurrent, aperiodic, irreducible M.C., $\lim_{n \rightarrow \infty} P_{ij}^n = \pi_j$ exists $\forall i, j \in S$ and π is the unique stationary distn for the M.C. In fact, for each state j ,

$$\pi_j = \frac{1}{E_j[T_j]} = \frac{1}{\text{mean return time to } j}$$

What about periodic, null recurrent/transient?

Assume MC is irreducible.

Trans. or null rec.: no stationary distribution, $\lim_{n \rightarrow \infty} P_{ij}^n = 0$ for all i, j .

positive rec.: there is a unique stationary distribution π . We have that for each i , a.s. under $P_i = P(\cdot | X_0 = i)$, for each j ,

$$\lim_{n \rightarrow \infty} \frac{1}{n+1} \sum_{k=0}^n 1_{\{X_k=j\}} = \lim_{n \rightarrow \infty} \frac{1}{n+1} \sum_{k=0}^n P_{ij}^k = \pi_j.$$

If the MC is aperiodic, $\lim_{n \rightarrow \infty} P_{ij}^n = \pi_j$ for all i and j . If the MC is periodic with period d , $\lim_{n \rightarrow \infty} P_{ii}^{nd} = d\pi_i$ for all i .

Idea of part of the proof of Basic Limit Thm.

Assume: irreducible, positive recurrent MC X .

We will establish a limit theorem for the Cesaro averages: $\frac{1}{n+1} \sum_{k=0}^n P_{ij}^k$.

Fix a state i as the starting state and another state j . All a.s. statements below are under the probability measure $P(\cdot | X_0 = i)$.

Let $T_{ij} = \inf\{n \geq 0 : X_n = j\}$ = the first time MC reaches j from i .
Let $T_j^{(k)}$ = time to return to j for the k -th time.

Using the strong law of large numbers: a.s. as $m \rightarrow \infty$,

$$\frac{1}{m} \sum_{k=1}^m T_j^{(k)} \longrightarrow E_i[T_j^{(1)}] \equiv m_j.$$

Note that $m_j < \infty$ since we are assuming positive recurrence.

Let $N_n = \sum_{k=0}^n 1_{\{X_k=j\}}$, the number of visits to j by time n .

Let

$$\begin{aligned} W_m &= T_{ij} + \sum_{k=1}^m T_j^{(k)} \\ &= \text{waiting time until } m\text{-th return to } j \end{aligned}$$

For convenience, we define $W_{-1} = 0$.

Note that

$$W_{N_n-1} \leq n < W_{N_n}.$$

That is, the time of the last visit to j before or at time n is W_{N_n-1} (if there is no prior visit to j , this random variable is zero, by our convention above). This arises because N_n is the number of visits to j by time n and $N_n - 1$ is the number of *returns* to j by time n (interpreting a value of -1 as meaning there are no returns and not even a first visit to j by time n). So W_{N_n-1} is the waiting time until the $(N_n - 1)$ -th return to j i.e., until the N_n -th visit to j . Furthermore, W_{N_n} is the waiting time until the N_n -th return to j or until the $(N_n + 1)$ -th visit to j and this time will be after n .

Dividing the above inequality through by N_n yields (assuming n is large enough that $N_n > 0$):

$$\frac{W_{N_n-1}}{N_n} \leq \frac{n}{N_n} < \frac{W_{N_n}}{N_n}$$

Now, by the assumed recurrence, a.s., $N_n \rightarrow \infty$ as $n \rightarrow \infty$, and combining this with the result of the strong law of large numbers derived

above, we conclude that a.s. as $n \rightarrow \infty$,

$$\frac{W_{N_n}}{N_n} \rightarrow m_j$$

and

$$\frac{W_{N_{n-1}}}{N_n} \rightarrow m_j.$$

Using the sandwich inequality above, it follows that a.s. as $n \rightarrow \infty$,

$$\frac{n}{N_n} \rightarrow m_j.$$

Since $0 < m_j < \infty$, it follows that a.s. as $n \rightarrow \infty$,

$$\frac{1}{n}N_n = \frac{1}{n} \sum_{k=0}^n 1_{\{X_k=j\}} \longrightarrow \frac{1}{m_j}.$$

It follows that a.s. as $n \rightarrow \infty$,

$$\frac{1}{n+1}N_n \longrightarrow \frac{1}{m_j}.$$

Furthermore, since the quantity on the left above is bounded for all n (by one), it follows from the bounded convergence theorem that as $n \rightarrow \infty$,

$$E_i \left[\frac{1}{n+1} \sum_{k=0}^n 1_{\{X_k=j\}} \right] \longrightarrow \frac{1}{m_j}.$$

In other words, as $n \rightarrow \infty$,

$$\frac{1}{n+1} \sum_{k=0}^n P_{ij}^k \longrightarrow \frac{1}{m_j}.$$