

Lecture 5

4-13-05

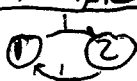
Long Run Behavior of Markov Chains

Assume $\{X_n\}_{n=1}^{\infty}$ is a Markov Chain with transition probability matrix $P = (P_{ij})$ and state space $S = \{0, 1, 2, 3, \dots\}$

Def: Steady state (or limiting) probability distribution is a probability vector $\pi = (\pi_i, i \in S)$ if $\lim_{n \rightarrow \infty} P_{ij}^n = \pi_j$ for all $i, j \in S$

- (1) $\lim_{n \rightarrow \infty} P_{ij}^n$ exists for each pair of i, j
- (2) $\lim_{n \rightarrow \infty} P_{ij}^n$ doesn't depend on initial state
- (3) $\sum_{j \in S} \pi_j = 1$

Example 1



$$P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$P_{11}^2 = P(X_2=1 | X_0=1) = 1$$

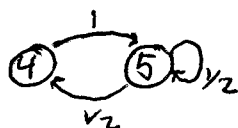
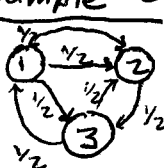
$$P_{11}^{2n} = P(X_{2n}=1 | X_0=1) = 1$$

$$P_{11}^1 = P(X_1=1 | X_0=1) = 0$$

$$P_{11}^{2n+1} = 0$$

Violates 1st condition

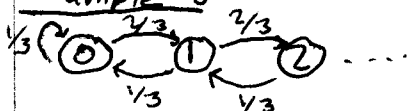
Example 2



$$\lim_{n \rightarrow \infty} P_{11}^n = \frac{1}{3} \quad \lim_{n \rightarrow \infty} P_{41}^n = 0$$

Violates 2nd condition

Example 3



$$\lim_{n \rightarrow \infty} P_{ij}^n = 0 \quad \forall i, j \in S$$

Violates 3rd condition

Defn: Stationary Distribution

A probability distribution $\pi = (\pi_i, i \in S)$ is called a stationary distribution iff $\pi P = \pi$, i.e. $\sum_{i \in S} \pi_i P_{ij} = \pi_j \quad \forall j \in S$

Lemma: Any steady state distribution is a stationary distribution

proof: Fix $j \in S$

$$\pi_j = \lim_{n \rightarrow \infty} P_{kj}^n \quad \forall k \in S$$

$$= \lim_{n \rightarrow \infty} \sum_{i \in S} P_{ki}^n P_{ij}$$

$$\geq \lim_{n \rightarrow \infty} \sum_{i \in S_m} P_{ki}^n P_{ij}$$

Where $\{S_m\}_{m=1}^{\infty}$ is a sequence of subsets of S such that $|S_m| \leq m$

$$P_{kj}^{n+1} = P(X_{n+1}=j | X_0=k)$$

$$= \sum_{i \in S} P(X_{n+1}=j, X_n=i | X_0=k)$$

$$= \sum_{i \in S} P(X_{n+1}=j | X_0=k, X_n=i) P(X_n=i | X_0=k)$$

$$= \sum_{i \in S} P(X_{n+1}=j | X_n=i) \underbrace{P(X_n=i | X_0=k)}_{P_{ki}^n}$$

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$$\pi_j \geq \sum_{i \in S} \pi_i p_{ij} \quad \forall m > 0$$

Let $m \rightarrow \infty$, $\pi_j \geq \sum_{i \in S} \pi_i p_{ij} \quad \forall j \in S$

Suppose $\exists j \in S$ where $\pi_{j_0} > \sum_{i \in S} \pi_i p_{ij_0}$ if $a_{m,n} \geq 0$

$$1 = \sum_{j \in S} \pi_j = \sum_{j \neq j_0} \pi_j + \pi_{j_0}$$

$$\geq \sum_{j \neq j_0} \sum_{i \in S} \pi_i p_{ij} + \pi_{j_0}$$

$$> \sum_{j \neq j_0} \sum_{i \in S} \pi_i p_{ij} + \sum_{i \in S} \pi_i p_{ij_0} \quad \sum_{j \in S} p_{ij} = 1$$

$$= \sum_{j \in S} \sum_{i \in S} \pi_i p_{ij} = \sum_{i \in S} \sum_{j \in S} p_{ij} \pi_i = \sum_{i \in S} \pi_i = 1$$

|x|

Warning: May have a stationary dist. with no steady state dist.

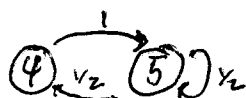
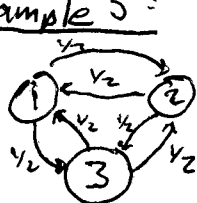
Example 4:



$$P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (\pi_1, \pi_2) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \pi_1 \\ \pi_2 \end{bmatrix}$$

stationary distribution $\rightarrow (\frac{1}{2}, \frac{1}{2}) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$
 but not steady state

Example 5:



$$\pi = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0)$$

$$\pi = (0, 0, 0, \frac{1}{3}, \frac{2}{3})$$

$$\begin{bmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1/2 & 1/2 \end{bmatrix}$$

Classification of States

① communicating classes

 $i \rightarrow j$: i leads to j iff $p_{ij}^n > 0$ for some $n \geq 0$ $P^0 = I, P_{ii}^0 = 1$
 $i \leftrightarrow j$: i communicates with j iff i leads to j and j leads to i

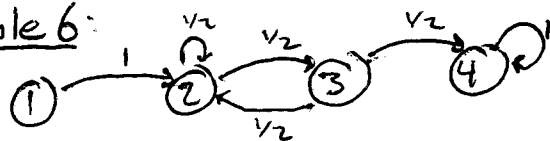
communication is an equivalence relation $i \leftrightarrow i, i \leftrightarrow j \Rightarrow j \leftrightarrow i$
 $i \leftrightarrow j, j \leftrightarrow k \Rightarrow i \leftrightarrow k$

 i leads to j , j leads to $k \Rightarrow i$ leads to k

Can partition states into disjoint communicating classes

$$S = \bigcup_{i \in S} C_i \quad C_i = \{j \in S : i \leftrightarrow j\}$$

Example 6:

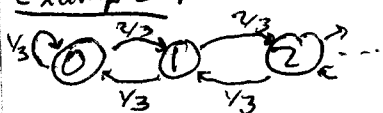


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 $\{1\}, \{2, 3\}, \{4\}$

Definition A Markov chain is irreducible if all of its states belong to a single communicating class.

Example 7:



irreducible, can get to any state from any state

Sufficient condition for irreducibility is $P^n > 0$ for some $n \geq 0$
 $P_{ij}^n > 0$ for all $i, j \in S$

Definition $d(i)$ is the period of the state i if $d(i)$ is the greatest common division of all $n \geq 1$ such that $P_{ii}^n > 0$
 (if $P_{ii}^n > 0$ for all $n \geq 1$ defined $d(i) = 0$)
 $P_{ii}^n > 0$