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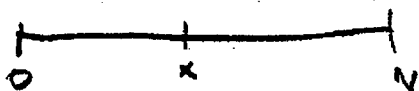
MATH 285A

Lecture 4

①

Hitting Times (cont.)

$$X_n = x + \sum_{i=1}^n \xi_i \quad \begin{aligned} P(\xi_i = 1) &= p \in (0, 1) \\ P(\xi_i = -1) &= q = 1 - p \end{aligned}$$



$$T_{0N} = \inf \{n \geq 0 : X_n = 0 \text{ or } N\}$$

Last Time

(i) $T_{0N} < \infty$ a.s. under $P_i = P(\cdot | X_0 = i)$, $i = 0, 1, \dots, N$

(ii) $E_i(T_{0N}) < \infty \quad \forall i = 0, 1, \dots, N$
 $(= \int_0^\infty T_{0N} dP_i)$

(iii) Compute

$$P_i(X_{T_{0N}} = 0) \quad i = 0, 1, \dots, N$$

$$E_i(T_{0N})$$

First Step Analysis

Let $\alpha_i = P_i(X_{T_{0N}} = 0)$ $i = 0, 1, \dots, N$ (probability you'll exit on the left endpoint)

and $\beta_i = E_i(T_{0N})$ $i = 0, 1, \dots, N$

$$\alpha = (\alpha_0, \dots, \alpha_N)$$

$$\alpha(0) = 1, \alpha(N) = 0$$

$$\alpha(i) = \alpha_i$$

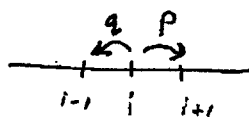
Fix $i \in \{1, \dots, N-1\}$

(2)

$$\begin{aligned}
 \alpha(i) &= P(X_{T_{ON}} = 0 \mid X_0 = i) = \sum_j P(X_{T_{ON}} = 0, X_1 = j \mid X_0 = i) \\
 &= \sum_j P(X_{T_{ON}} = 0 \mid X_1 = j, X_0 = i) P(X_1 = j \mid X_0 = i) \quad \text{by Bayes Rule} \\
 &= \sum_j P(X_{T_{ON}} = 0 \mid X_0 = j) P_{ij} \quad \text{by Markov Property and Time Homogeneity} \\
 &= \sum_j \alpha(j) P_{ij} \\
 &= p\alpha(i+1) + q\alpha(i-1)
 \end{aligned}$$

So

$$q\alpha(i-1) - \alpha(i) + p\alpha(i+1) = 0 \quad i \in \{1, \dots, N-1\}$$

So $A\alpha = 0$ where A is a $(N-1) \times (N+1)$ matrix

$$A = \begin{pmatrix} q & -1 & p & 0 & \dots & 0 \\ 0 & q & -1 & p & 0 & \dots & 0 \\ 0 & 0 & q & -1 & p & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & q & -1 & p \end{pmatrix}$$

with linearly independent rows

- since A is $(N-1) \times (N+1)$ there are 2 linearly independent solutions α_1, α_2 of $A\alpha = 0$. (Rank $A = N-1$)

Try $\alpha(i) = \gamma^i$ for some $\gamma > 0$
 then we need:

$$q\gamma^{i-1} - \gamma^i + p\gamma^{i+1} = 0$$

$$\Leftrightarrow q - \gamma + p\gamma^2 = 0$$

$$\Leftrightarrow (p\gamma - q)(\gamma - 1) = 0$$

$$\Leftrightarrow \gamma = \frac{q}{p} \text{ or } \gamma = 1.$$

③

If $q \neq p$,

General solution of $Aa = 0$ is

$$a(i) = c_1 \left(\frac{q}{p}\right)^i + c_2(1), \quad i = 0, 1, \dots, N$$

$$a(0) = 1 \Rightarrow 1 = c_1 + c_2$$

$$a(N) = 0 \Rightarrow 0 = c_1 \left(\frac{q}{p}\right)^N + c_2$$

so

$$a(i) = \frac{1}{1 - \left(\frac{q}{p}\right)^N} \left(\left(\frac{q}{p}\right)^i - \left(\frac{q}{p}\right)^N \right) \quad i = 0, \dots, N$$

If $q = p$

general solution is

$$a(i) = c_1 + c_2 i$$

$$a(0) = 1 \quad \text{so} \quad 1 = c_1$$

$$a(N) = 0 \quad \text{so} \quad 0 = c_1 + c_2 N$$

so

$$a(i) = 1 - \frac{i}{N} \quad i = 0, \dots, N$$

β

$$\beta(i) = E_i(T_{0N})$$

$$\beta(0) = 0 = \beta(N)$$

(Note: we know from last time that $\beta(i)$ is finite for all i)

Fix $i \in \{1, \dots, N-1\}$

$$\beta(i) = E_i(T_{0N}) = E(T_{0N} | X_0 = i)$$

$$= \sum_j E(T_{0N} 1_{[X_1=j]} | X_0 = i)$$

$$= \sum_j E(T_{0N} | X_1 = j, X_0 = i) P(X_1 = j | X_0 = i)$$

$$= \sum_j (1 + E(T_{0N} | X_0 = j)) p_{ij} \quad \text{by Markov Property and shifting time}$$

④

$$= 1 + \sum_j \underbrace{E(T_{0N} | X_0 = j)}_{\beta(j)} P_{ij}$$

$$\text{So } \beta(i) = 1 + \sum_j P_{ij} \beta(j) \quad \text{for } i = 1, \dots, N-1$$

$$\text{So } A\beta = -1 \quad \sum_j P_{ij} \beta(j) - \beta(i) = -1$$

$$A\alpha = 0$$

General solution

$$\beta(i) = C_1 \alpha_1(i) + C_2 \alpha_2(i) + \beta_1(i)$$

where β_1 is a particular solution of $A\beta = -1$
and α_1, α_2 are linearly indep. solns of $A\alpha = 0$ (homogeneous eqn)

For $p \neq q$ ($p \neq \frac{1}{2}$)

Seek partic. solution β ,

$$\beta_1(i) = p\beta_1(i+1) + q\beta_1(i-1) + 1$$

$$\alpha_1(i) = \left(\frac{q}{p}\right)^i, \quad \alpha_2(i) = 1$$

Try solution: $\beta_1(i) = c_3 i$

$$c_3 i = p c_3 (i+1) + q c_3 (i-1) + 1$$

$$c_3 = \frac{1}{q-p} = \frac{1}{1-2p}$$

General solution: ($p \neq q$)

$$\beta(i) = c_1 \left(\frac{q}{p}\right)^i + c_2 + \frac{i}{1-2p}$$

$$\beta(0) = 0 \Rightarrow 0 = c_1 + c_2$$

$$\beta(N) = 0 \Rightarrow 0 = c_1 \left(\frac{q}{p}\right)^N + c_2 + \frac{N}{1-2p}$$

$$\beta(i) = \frac{N}{(1-\frac{q}{p})^N (1-2p)} \left(\left(\frac{q}{p}\right)^i - 1 \right) + \frac{i}{1-2p}$$

$\forall i$

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For $p=q=\frac{1}{2}$

$$\beta(i) = c_1 + c_2 i + \beta_1(i)$$

Try $\beta_1(i) = c_3 i^2$

Need: $c_3 i^2 = 1 + \frac{1}{2} c_3 (i+1)^2 + \frac{1}{2} c_3 (i-1)^2$

$$\Leftrightarrow 0 = 2 + 2c_3 \Rightarrow c_3 = -1$$

solution:

$$\beta(i) = c_1 + c_2 i - i^2$$

$$\beta(0) = \beta(N) = 0, \text{ so } c_1 = 0, c_2 = N$$

$$\beta(i) = Ni - i^2 = (N-i)i$$

What Happens when $N \rightarrow \infty$?

$$T_{0N} \rightarrow T_0 = \inf\{n \geq 0 : X_n = 0\} \text{ as } N \rightarrow \infty$$

$$P_i(X_{T_{0N}} = 0) = \begin{cases} \left(\frac{1 - (\frac{q}{p})^N}{1 - (\frac{q}{p})} \right) \left(\left(\frac{q}{p} \right)^i - \left(\frac{q}{p} \right)^N \right), & p \neq q \\ 1 - \frac{i}{N}, & p = q \end{cases}$$

$$E_i(T_{0N}) = \begin{cases} \frac{N}{(1 - (\frac{q}{p})^N)(1 - 2p)} \left(\left(\frac{q}{p} \right)^i - 1 \right) + \frac{i}{1 - 2p}, & p \neq q \\ Ni - i^2, & p = q \end{cases}$$

(6)

$$\{X_{T_{0N}} = 0\} = \{\text{hit } 0 \text{ before } N\}$$

$$\{\text{hit } 0 \text{ ever}\} = \bigcup_N \{\text{hit } 0 \text{ before } N\} \quad (\{\text{hit } 0 \text{ before } N\} \subset \{\text{hit } 0 \text{ before } N+1\})$$

$$\Rightarrow P_i(\text{hit } 0 \text{ ever}) = \lim_{N \rightarrow \infty} \uparrow P_i(\text{hit } 0 \text{ before } N)$$

$$= \lim_{N \rightarrow \infty} \uparrow P_i(X_{T_{0N}} = 0)$$

$$= \begin{cases} 1 & \text{if } p = q \\ \left(\frac{q}{p}\right)^i & \text{if } p > q \quad \left(\frac{q}{p} < 1\right) \\ 1 & \text{if } p < q \end{cases}$$

$$T_{0N} \uparrow T_0 \text{ as } N \rightarrow \infty$$

So by Monotone Convergence Theorem,

$$E_i(T_0) = \lim_{N \rightarrow \infty} E_i(T_{0N})$$

$$= \begin{cases} +\infty & q = p \\ +\infty & q < p \quad \left(\frac{q}{p} < 1\right) \\ \frac{1}{1-2p} & p < q \quad \left(\frac{q}{p} > 1\right) \end{cases}$$